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Short distance properties of cascading gauge theories

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Abstract

We study the short distance (large momentum) properties of correlation functions of cascading gauge theories by performing a tree-level computation in their dual gravitational background. We prove that these theories are holographically renormalizable; the correlators have only analytic ultraviolet divergences, which may be removed by appropriate local counterterms. We find that n -point correlation functions of properly normalized operators have the expected scaling in the semi-classical gravity (large N) limit: they scale as N_{eff}^{2-n} with $N_{eff} \propto \ln(k/\Lambda)$ where k is a typical momentum. Our analysis thus confirms the interpretation of the cascading gauge theories as renormalizable four-dimensional quantum field theories with an effective number of degrees of freedom which logarithmically increases with the energy.

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1 Introduction and summary

Cascading gauge theories were discovered in [1–3] (see [4, 5] for reviews) by looking at the decoupling limit (the near-horizon limit) of fractional D3-branes at a conifold singularity. Since then various other examples have also been studied, including [6–13]. These theories are not standard local quantum field theories since they do not approach a conformal field theory at high energies. Therefore, one cannot use standard field theory techniques to analyze them. When one introduces a finite high-energy cutoff at some scale M , then at the cutoff scale these theories resemble an $\mathcal{N} = 1$ supersymmetric $SU(K) \times SU(K+P)$ gauge theory with two bifundamental and two anti-bifundamental chiral superfields and some superpotential. When one flows down in energy from this cutoff one of the gauge groups becomes strongly coupled and the theory seems to undergo a series of Seiberg duality [14] “cascades”, reducing the value of K , and finally ending (when K is a multiple of P) with a confining theory at some low-energy scale Λ [3, 5, 15] (which is related to the $\mathcal{N} = 1$ supersymmetric pure $SU(P)$ Yang-Mills theory). However, the value of K increases with the high-energy (UV) cutoff as $K \propto \ln(M/\Lambda)$, so it seems that an increasing number of degrees of freedom is needed to define the theory at higher energies, and that the ultimate definition of the cascading theory requires a theory with an infinite number of fields.

It is not known how to directly define a cascading gauge theory in field theory terms¹. The best available definition of the cascading gauge theory is via its holographic dual background [1–3]. This background can be well-described by a semi-classical supergravity theory when the dimensionless parameter of the cascading gauge theories, $g_s P$ (where g_s is the string coupling in the dual background), is large. However, the asymptotic region of this background is actually well-described by supergravity for any value of this dimensionless parameter, related to the fact that the effective ’t Hooft coupling constant, $g_{YM}^2 K$, always becomes large in these theories at high energies. Since the computations of short-distance correlation functions that we will perform will be dominated by this asymptotic region, our results will be valid (at short enough distances) for any value of $g_s P$.

In the AdS/CFT correspondence [17–19] (see [20] for a review) properties of the conformal field theory (CFT) may be computed using its holographic dual theory on

¹It may be possible to define it by a limiting procedure, using an infinite-number-of-fields-limit of well-defined theories which flow to the cascade at some energy scale, as in the construction of [16].

anti-de Sitter (AdS) space. In particular, many computations can be done when the supergravity approximation is valid. The same is true also for the cascading gauge theories [21–24]. It was shown in [23] that despite having an infinite number of high-energy degrees of freedom, all one-point functions of the cascading gauge theory (including the conformal anomaly) are finite after they are holographically renormalized. The consistency of this renormalization procedure was tested in [25]. It can be used to compute the thermodynamic properties, such as the pressure and the energy density, of the cascading gauge theory plasma, by evaluating the one-point function of the stress-energy tensor. From these one can compute the speed of sound in the theory. The same speed of sound is precisely reproduced from the dispersion relation for the sound waves extracted from the pole in the stress-energy tensor two-point correlation function, which can be evaluated without holographic renormalization.

In this paper we continue exploring the holographic definition of cascading gauge theories. We compute the large-momentum limit of a specific contribution to the N -point functions of operators dual to supergravity fields. We show that this contribution has only analytic ultraviolet divergences (contact terms) and, therefore, it may be renormalized with local counterterms. We demonstrate that holographic renormalizability of the N -point correlation functions in the cascading theories is directly linked to the renormalizability of the corresponding conformal field theories (which arise in the $P \rightarrow 0$ limit). For example, this implies that the renormalizability of the Klebanov-Strassler cascading theory [2, 3] follows from the renormalizability of the Klebanov-Witten supersymmetric CFT [26]. Moreover, we find that properly normalized cascading gauge theory operators have the expected scaling of their N -point correlation functions. They behave as N_{eff}^{2-N} with $N_{eff} \propto \ln(k/\Lambda)$, where k is a typical momentum scale, as expected in the 't Hooft large N_{eff} limit [27] of an $SU(N_{eff})$ gauge theory (or an $SU(N_{eff}) \times SU(N_{eff} + P)$ gauge theory). Our analysis confirms the interpretation of the cascading gauge theories as renormalizable four-dimensional quantum field theories whose effective number of degrees of freedom logarithmically increases with the energy, as suggested also by their thermodynamic behavior at high temperatures [28–30].

This paper is organized as follows. We begin in section 2 by outlining the computation that we need to perform. In section 3 we compute the bulk-to-boundary propagator in the cascading background, and use it to compute the two-point functions of the cascading theories (some specific cases of two-point functions were previously computed

in [21, 22]). The main computation of the N -point correlation functions is performed in section 4, and our main results appear in §4.2.4. Two appendices contain technical details.

Our analysis leaves some remaining open problems. First, we were only able to compute the large-momentum limit of a specific class of correlation functions. It would be interesting to find a way to generalize our computation to arbitrary correlation functions. It would also be interesting to extend our results to finite values of the momentum. Such correlation functions depend on the IR behavior of the cascading theory, which our large-momentum computation is independent of. In particular, the computation of correlators at finite momentum is necessary in order to compute the S-matrix of the cascading theories (through an LSZ-type procedure), and without it we are not able to say how the S-matrix of these theories behaves at high energies.

Second, we only discuss here correlation functions of the cascading gauge theory operators which are dual to ten dimensional supergravity modes². It would be very interesting to evaluate also the correlation functions of the operators dual to massive string modes, and to verify whether or not they confirm the picture of the cascading theory as a renormalizable quantum field theory with $N_{eff} \propto \ln(k/\Lambda)$ effective degrees of freedom. Note that this computation is complicated since the anomalous dimension of the corresponding operators seems to grow as $(g_{YM}^2 N_{eff})^{1/4}$, which grows without bound as the energy is increased. A first important step in this direction was made in [31] where it was shown that the anomalous dimension of twist-2 operators in cascading gauge theories has the expected dependence on N_{eff} .

2 Generalities

We would like to study the high momentum correlation functions of cascading gauge theories. Specifically, we look at the cascading gauge theory of fractional D3-branes at a conifold singularity, whose gravitational dual is asymptotically given by the Klebanov-Tseytlin (KT) [2] solution of type IIB supergravity³

$$\frac{1}{L^2} ds_{10}^2 = g_{\mu\nu} dy^\mu dy^\nu + \sqrt{h} ds_{T^{1,1}}^2 = \frac{1}{\sqrt{h}} \frac{1}{\rho^2} dx^{i2} + \frac{\sqrt{h}}{\rho^2} d\rho^2 + \sqrt{h} ds_{T^{1,1}}^2, \quad (2.1)$$

²This includes in particular the operators dual to the specific Kaluza-Klein modes appearing in the effective 5d supergravity description of the asymptotically cascading geometries, obtained in [23].

³There are also some p -form fields turned on in the background, which will not play any role in our computations.

where $h(\rho) = \frac{1}{8}P^2p_0 + \frac{1}{4}K_0 - \frac{1}{2}P^2p_0 \ln(\rho/\rho_0)$ in the notation of [23], $\mu, \nu = 0, 1, 2, 3, 4$, $i = 0, 1, 2, 3$, p_0 is the string coupling and P is the number of fractional D3-branes. We note that the details of this background (such as the $T^{1,1}$ metric) will not play any role in our computations. These should be valid for any cascading gauge theory background which has a similar logarithmic form of the warp factor $h(\rho)$. In order to keep our discussion general we will denote $h(\rho) = a' - b \ln(\rho/\rho_0)$. The AdS background is then a special case of this with $b = 0$.

As usual, we will perform a Kaluza-Klein (KK) reduction of all the fields on the compact space $T^{1,1}$. After this reduction we obtain an infinite set of five dimensional fields $\phi_i(x, \rho)$, which are dual to operators $\mathcal{O}_i(x)$ in the dual field theory.

In the AdS/CFT correspondence [17–19] and its generalizations, one obtains the field theory correlation functions $\langle \mathcal{O}_1(x_1) \dots \mathcal{O}_N(x_N) \rangle$ in the gravity approximation by varying the gravity action with respect to the boundary value of the fields ϕ_i dual to $\mathcal{O}_i(x_i)$. One contribution to such a correlation function will come from an interaction vertex in the bulk action (if it exists) of the form $\lambda_N \phi_1 \dots \phi_N$.

A simple way to compute the correlation functions is to define a bulk-to-boundary (BtB) propagator for the fields ϕ_i , denoted $K_i(x, x'; \rho)$, which gives the change in the field $\phi_i(x', \rho)$ in response to a (appropriately normalized) delta function source at the point x on the boundary ($\rho \rightarrow 0$). In section 3 we will review the form of these BtB propagators in AdS space, and compute them for the cascading background.

If we have a vertex of the form $\lambda_N \phi_1 \dots \phi_N$ in the five dimensional effective action, we obtain by the usual Feynman diagram techniques a contribution to the correlation function of the form

$$\langle \mathcal{O}_1(x_1) \dots \mathcal{O}_N(x_N) \rangle = \lambda_N \int \left(\prod_{i=1}^N K_i(x_i, x; \rho) \right) h^{5/4}(\rho) \sqrt{g} d^4x d\rho, \quad (2.2)$$

where g is the five dimensional metric appearing in (2.1) and the factor $h^{5/4}$ comes from the determinant of the $T^{1,1}$ metric. We have absorbed some normalization factors (including the overall scale L) into the variation of the action with respect to the boundary values of the fields.

A five dimensional interaction vertex of this form would arise from an interaction between N scalar fields in ten dimensions (with ϕ_i the KK modes of these scalar fields). However, there are no such interactions in the 10d supergravity (SUGRA) action. All the interactions in the 10d action involve two derivatives. Namely, in 10d they look like a product of two derivatives of 10d fields times a product of other fields. When

we reduce to 5d, we can get non-derivative interactions of the type described above from ten-dimensional interactions with the derivatives in the angular ($T^{1,1}$) directions; however, we then get an additional factor of $h^{-1/2}(\rho)$ in the interaction vertex, coming from the metric in the angular coordinates. Thus, the non-derivative interactions in the 5d SUGRA action which actually exist take the form $\lambda_N h^{-1/2}(\rho) \phi_1 \cdots \phi_N$, and their contribution to correlation functions takes the form

$$\langle \mathcal{O}_1(x_1) \cdots \mathcal{O}_N(x_N) \rangle = \lambda_N \int \left(\prod_{i=1}^N K_i(x_i, x; \rho) \right) h^{3/4}(\rho) \sqrt{g} d^4 x d\rho. \quad (2.3)$$

Next, using

$$\sqrt{g} = \rho^{-5} h^{-3/4}, \quad (2.4)$$

and Fourier transforming both sides of (2.3), we find that the momentum space correlation functions may be written using the momentum space BtB propagators \hat{K}_i (using translational and rotational symmetry, which implies that the BtB propagators only depend on the absolute value of their momentum) in the form

$$\langle \hat{\mathcal{O}}_1(\vec{k}_1) \cdots \hat{\mathcal{O}}_N(\vec{k}_N) \rangle = \delta(\sum_{i=1}^N \vec{k}_i) \int \prod_{i=1}^N \hat{K}_i(|\vec{k}_i|; \rho) \rho^{-5} d\rho, \quad (2.5)$$

where $\hat{\mathcal{O}}_i(\vec{k}_i)$ is the Fourier transform of $\mathcal{O}_i(x_i)$. The large k behavior can then be extracted from this. We concentrate in this paper on non-derivative interactions in five dimensions. Derivative interactions in the five dimensional effective action can be treated similarly (they give expressions similar to (2.5) but with ρ -derivatives of some of the propagators or with additional factors of k_i) and do not result in any qualitatively new physics.

The expression (2.5) is IR divergent as $\rho \rightarrow 0$ (this corresponds to a UV divergence in the field theory) and must be renormalized. To regulate the theory, we put a cutoff at some small (close to the boundary) radial coordinate ρ_{UV} , and define a regularized bulk-to-boundary propagator corresponding to a source at ρ_{UV} instead of at the boundary. In addition, the integral over the ρ coordinate in (2.5) extends from ρ_{UV} to infinity. Eventually, we need to take the $\rho_{UV} \rightarrow 0$ limit. Generically, the integral (2.5) diverges as ρ_{UV} goes to zero. However, we will show that all the divergent terms are analytic in (some of) the momenta, so they are contact terms in position space.

In the usual AdS/CFT correspondence, one can consistently subtract these divergences using holographic renormalization, by adding appropriate counter-terms to the

action. The final result for N -point functions is given by the non-analytic terms (in k) in the above expressions. These terms are non-divergent as $\rho_{UV} \rightarrow 0$ (if we are careful to take the $\rho_{UV} \rightarrow 0$ limit only at the very end of the calculation). We will use precisely the same regularization and renormalization method in the cascading background. We will find that this procedure leads to finite results for the N -point correlation functions, which are given by the non-analytic terms in (2.5) in the $\rho_{UV} \rightarrow 0$ limit.

3 Bulk-to-boundary propagators and two-point functions

In this section we compute the BtB propagator in the Klebanov-Tseytlin background [2] which is needed for the computation of correlation functions. We specialize to the case of a scalar field in five dimensions, coming from a KK reduction of some type IIB supergravity field on the $T^{1,1}$. We expect the generalization of our results to fields of higher spin to be straightforward. We will find it more convenient to work in momentum space rather than in position space. Since holographic correlation functions are usually computed in position space, we start in §3.1 by reviewing the computation of the momentum space BtB propagator in AdS space. This turns out to be useful because of the similarity between the KT and AdS backgrounds. In §3.2 we compute the BtB propagator in the KT background, and in §3.3 we use this for the computation of two-point functions.

3.1 The AdS bulk-to-boundary propagator

We would like to find the BtB propagator for a scalar field of mass m moving in the AdS_5 background. This background is given by setting $h = 1$ in (2.1), in which case the scale L becomes the radius of curvature (note that both x and ρ have units of length). We assume that this scalar arises as some KK mode on $T^{1,1}$ with the mass coming from the Laplacian in the $T^{1,1}$ directions. Plugging a solution with momentum k^i in the x^i directions into the equation of motion $(\square + m^2)\phi = 0$, we find the equation

$$\rho^2(\rho^{-2}\phi)'' + \rho(\rho^{-2}\phi)' - (4 + m^2L^2 + k^2\rho^2)(\rho^{-2}\phi) = 0. \quad (3.1)$$

Here primes denote derivatives with respect to ρ , and $k \equiv |\vec{k}|$.

Equation (3.1) is invariant under rescaling $\rho \rightarrow \alpha\rho$, $k \rightarrow k/\alpha$, so the non-trivial features of the solution will be at values of ρ of order $1/k$. Now, suppose that we solve the equation of motion in a space which is only asymptotically AdS, with significant

differences from AdS occurring at $\rho > \rho_0$ (where ρ_0 is the scale where IR effects become important). We expect that the solution to (3.1) will be a good approximation to the solution we are interested in as long as $k \gg 1/\rho_0$. The corrections to this solution will be a power series in $1/k\rho_0$, so our results will be valid at large enough momentum in any asymptotically AdS space.

To solve (3.1), we define $\psi \equiv \rho^{-2}\phi$ and switch to dimensionless variables $R = \rho/\rho_s$ and $Y = k\rho_s$, where ρ_s is some arbitrary scale which we introduce for convenience (it will, of course, drop out of all physical results). In these variables the equation of motion takes the form

$$R^2\psi''(R) + R\psi'(R) - (\nu^2 + Y^2R^2)\psi(R) = 0, \quad (3.2)$$

where $\nu^2 = 4 + m^2L^2$ is related to the dimension Δ of the dual operator through $\Delta = \nu + 2$. We restrict to $\nu > 0$ such that we are strictly above the Breitenlohner-Freedman bound for scalar fields in AdS space.

One method to solve the equation (3.2) is by an expansion at small R . In the small R limit there are two asymptotic solutions to the equation of motion, $\psi \sim R^{\pm\nu}$. One can then expand the solution as a power series in R and in $\ln(R)$ around $R \sim 0$. We find it convenient to write the two linearly independent solutions as

$$\psi_\nu(R) = R^\nu \sum_{n=0}^{\infty} \tilde{\kappa}_{\nu,n}(YR)^n \quad (3.3)$$

and

$$\psi_{-\nu}(R) = R^{-\nu} \sum_{n=0}^{\infty} \tilde{\kappa}_{-\nu,n}(YR)^n \quad (3.4)$$

for non-integer ν . For integer ν the second solution, equation (3.4), is replaced by

$$\psi_{-\nu}(R) = R^{-\nu} \sum_{n=0}^{2\nu-1} \tilde{\kappa}_{-\nu,n}(YR)^n + (-1)^{\nu+1} \ln(R) \psi_\nu(R). \quad (3.5)$$

Before we impose any boundary conditions, the general solution to the equations of motion is of the form

$$\phi = C_\nu(R^2\psi_{-\nu}(R) + \alpha_\nu R^2\psi_\nu(R)), \quad (3.6)$$

where C_ν and α_ν are arbitrary constants. The BtB propagator $K(Y, R)$ is defined to be the solution with boundary conditions such that it is finite in the interior, and such

that it equals $\rho_{UV}^{2-\nu}$ at the UV boundary $\rho = \rho_{UV}$ (this is the Fourier transform of the position-space boundary condition $K(x_0, x; \rho) \rightarrow \rho_{UV}^{2-\nu} \delta(x - x_0)$). C_ν is easily determined by the UV boundary condition: $C_\nu = \rho_{UV}^{2-\nu} (R_{UV}^2 \psi_{-\nu}(R_{UV}) + \alpha_\nu R_{UV}^2 \psi_\nu(R_{UV}))^{-1}$, with $R_{UV} \equiv \rho_{UV}/\rho_s$. On the other hand, α_ν is determined by the boundary condition in the interior of AdS. To find it, one needs some handle on the asymptotic behavior of the solutions $\psi_{\pm\nu}(R)$ at large R , which is not evident from the series expansions we wrote above which are useful only when $RY \ll 1$.

Fortunately, equation (3.2) is a Bessel equation and has been thoroughly studied. The solution which is smooth as $R \rightarrow \infty$ is given by a modified Bessel function of the second kind,

$$\hat{K}(Y, R) = \rho_{UV}^{2-\nu} \frac{R^2 K_\nu(RY)}{R_{UV}^2 K_\nu(R_{UV} Y)}. \quad (3.7)$$

This leads to ⁴

$$\alpha_\nu = \begin{cases} -Y^{2\nu} & \nu \text{ non-integer,} \\ (-1)^{\nu+1} Y^{2\nu} \ln(\frac{1}{2}Y) & \nu \text{ integer.} \end{cases} \quad (3.8)$$

For later convenience, we give here the coefficients of the small R expansion of the modified Bessel function. We will find it useful to write K_ν with a somewhat non-standard normalization. We write

$$K_\nu(y) = y^{-\nu} \left(\sum_{n=0}^{\nu-1} \kappa_{2n,0} y^{2n} + \sum_{n=\nu}^{\infty} \sum_{m=0}^1 \kappa_{2n,m} y^{2n} \ln^m(y) \right) \quad (3.9)$$

for integer ν , and

$$K_\nu(y) = y^{-\nu} \sum_{n=0}^{\infty} \kappa_{2n,0} y^{2n} + y^\nu \sum_{n=0}^{\infty} \kappa_{2\nu+2n,0} y^{2n} \quad (3.10)$$

for non-integer ν . We choose $\kappa_{0,0} = 1$. The other coefficients are given in the following table. Note that these κ 's are different from the $\tilde{\kappa}$'s in equations (3.3) and (3.4).

3.2 The KT bulk-to-boundary propagator

Next we wish to solve for the BtB propagator in the KT background (2.1), for a scalar field arising as a KK mode on $T^{1,1}$. We will use the same notation as in (3.2), where

⁴Note that the overall power of Y is obvious from dimensional analysis, as we know that the scale ρ_s should not appear in the solution, but this does not determine the coefficient.

Table 1: Coefficients of the modified Bessel function of the second kind.

ν integer	$n < \nu$	$\kappa_{2n,0} = \frac{(-1)^n \Gamma(\nu-n)}{2^{2n} \Gamma(n+1) \Gamma(\nu)}$
	$n \geq \nu$	$\kappa_{2n,1} = \frac{(-1)^{\nu+1}}{2^{2n-1} \Gamma(\nu) \Gamma(n-\nu+1) \Gamma(n+1)}$
		$\kappa_{2n,0} = \frac{(-1)^\nu (\psi(n-\nu+1) + \psi(n+1))}{2^{2n} \Gamma(n-\nu+1) \Gamma(n+1) \Gamma(\nu)} - \ln(2) \kappa_{2n,1}$
ν non-integer		$\kappa_{2n,0} = \frac{(-1)^n \Gamma(\nu-n)}{2^{2n} \Gamma(n+1) \Gamma(\nu)}$
		$\kappa_{2\nu+2n,0} = -\frac{\Gamma(1-\nu)}{2^{2n+2\nu} \Gamma(n+1) \Gamma(n+\nu+1)}$

now we can choose $\rho_s = 1/\Lambda$ to be a typical IR scale in the cascading geometry. The equation of motion that we find, again writing the solution as $\hat{K}(R) = R^2 \psi(R)$, is

$$R^2 \psi'' + R \psi' - (\nu^2 + Y^2 R^2 h(R)) \psi = 0, \quad (3.11)$$

where $h(R) \equiv a' - b \ln(R \rho_s / \rho_0) = a - b \ln(R)$ (with $a \equiv a' - b \ln(\rho_s / \rho_0)$). It will sometimes be convenient to choose the scale ρ_s such that $a = 0$. Note that for KK modes, whose mass comes from the Laplacian of some field on $T^{1,1}$, the factors of h in (2.1) conspire such that ν appears in the equation in exactly the same way as in AdS space. Our analysis is an extension of the specialized analysis of the $m^2 = 0$ case done in [21, 22].

Of course, since the background (2.1) is singular it is not really meaningful to solve the equation of motion in it. We are really interested in the solutions to the equations of motion in a regular space which asymptotes to (2.1), such as the backgrounds found in [3] or in [28–30]. As in the AdS case, we expect (and we can verify this based on our results) that at large momentum k , the dominant contributions to correlation functions will come from small values of ρ of order $1/k$ and that they will be independent of the IR (large ρ) resolution of the KT background. Thus, we will be interested in computing the large k limit of the BtB propagator and of the correlation functions, which is universal to all asymptotically KT backgrounds. The details of the IR resolution at a scale ρ_0 will affect corrections to the results of order $1/k\rho_0$.

As in the AdS case, we start by finding a small R expansion for the solution to (3.11), which we will denote by $K^{(I)}$. For non-integer values of ν , we write the solution as a sum $\psi(R) \sim \psi_\nu(R) + \psi_{-\nu}(R)$ of two linearly independent series expansions,

$$\psi_{-\nu}(R) = R^{-\nu} \sum_{\substack{n=0 \\ m \leq n}}^{\infty} p_{2n,m} R^{2n} \ln^m(R), \quad \psi_\nu(R) = R^\nu \sum_{\substack{n=0 \\ m \leq n}}^{\infty} p_{2n+2\nu,m} R^{2n} \ln^m(R). \quad (3.12)$$

For integer values of ν it is simpler to write the two independent solutions in a single series

$$\psi(R) = R^{-\nu} \left\{ \sum_{n=0}^{\nu-1} \sum_{m=0}^n p_{2n,m} R^{2n} \ln^m(R) + \sum_{n=\nu}^{\infty} \sum_{m=0}^{n+1} p_{2n,m} R^{2n} \ln^m(R) \right\}. \quad (3.13)$$

Plugging these ansatze into the equation of motion (3.11), one obtains a recursive relation for the coefficients $p_{2n,m}$ which is written and solved in appendix A. One finds that, both for integer and for non-integer ν , two of the coefficients are undetermined. One can choose these to be $p_{0,0}$ and $p_{2\nu,0}$. This behavior is analogous to the AdS case (see equations (3.3) - (3.5)), where we denoted the undetermined constants by C_ν and α_ν . The coefficients of the leading power of $\ln(R)$, appearing at each order in the expansion of the solution in powers of R , are summarized in table 2.

Table 2: Leading coefficients in the expansions (3.12) and (3.13). See (A.13)-(A.17) for integer ν , and (A.6)-(A.8) for non-integer ν . The expression for the integer ν , $n = \nu + s$ case includes only the dependence of $p_{2\nu+2s,s}$ on $p_{2\nu,0}$ and does not include its dependence on $p_{0,0}$.

ν integer	$1 \leq n < \nu$	$p_{2n,n} = \frac{(bY^2)^n \Gamma(\nu-n)}{2^{2n} \Gamma(n+1) \Gamma(\nu)} p_{0,0}$
	$n = \nu$	$p_{2\nu,\nu+1} = -\frac{(bY^2)^\nu}{2^{2\nu-1} \Gamma(\nu) \Gamma(\nu+2)} p_{0,0}$
	$n > \nu$	$p_{2n,n+1} = \frac{(-1)^{n+\nu+1} (bY^2)^n}{2^{2n-1} (\nu+1) \Gamma(\nu) \Gamma(n-\nu+1) \Gamma(n+1)} p_{0,0}$
	$n = \nu + s$	$p_{2\nu+2s,s} \cong \frac{(-bY^2)^s \Gamma(\nu+1)}{2^{2s} \Gamma(s+1) \Gamma(s+\nu+1)} p_{2\nu,0}$
ν non-integer	$n \geq 1$	$p_{2n,n} = \frac{(bY^2)^n \Gamma(\nu-n)}{2^{2n} \Gamma(n+1) \Gamma(\nu)} p_{0,0}$
	$n \geq 1$	$p_{2\nu+2n,n} = \frac{(-bY^2)^n \Gamma(1+\nu)}{2^{2n} \Gamma(n+1) \Gamma(n+1+\nu)} p_{2\nu,0}$

To find the BtB propagator we need to determine the two integration constants. One of the constants is determined from the UV boundary condition which we choose to be the same as in AdS,

$$\hat{K}^{(I)}(R_{UV}) = R^2 \psi(R) \Big|_{R=R_{UV}} = \rho_{UV}^{2-\nu}, \quad (3.14)$$

while the other one is fixed by requiring that the propagator is non-singular everywhere in the interior. To find the latter coefficient, we need some handle on the asymptotic

behavior of the solutions as ρ becomes very large. This cannot be obtained from the perturbative expansion we have given here, as this expansion is valid only in the region (which we call region I) where

$$Y^2 R^2 \ln(R) \ll 1. \quad (3.15)$$

In order to find the correct integration constants we will use the method of Krasnitz [21, 22], which is to solve the equations of motion in a region which allows for an evaluation of the asymptotic value of the field and also has some overlap with region I (3.15).

Consider the following approximation:

$$h(R) = a - b \ln\left(\frac{RY}{Y}\right) = a + b \ln(Y) \left(1 - \frac{\ln(RY)}{\ln(Y)}\right) = h_Y \left(1 - \frac{b \ln(RY)}{h_Y}\right), \quad (3.16)$$

where we defined $h_Y \equiv h\left(\frac{1}{Y}\right) = a + b \ln(Y)$. We would like to solve the equation of motion (3.11) in a region where one has $h(R) \simeq h_Y$. We require

$$|b \ln(RY)| \ll h_Y, \quad (3.17)$$

which for large momentum, $Y \gg 1$, means

$$|\ln(RY)| \ll \ln(Y). \quad (3.18)$$

In this region (which we call region II) we may approximate the equation of motion (3.11) as

$$R^2 \psi'' + R \psi' - (\nu^2 + R^2 Y^2 h_Y) \psi = 0 \quad (3.19)$$

which is simply a Bessel equation. This Bessel equation has two independent solutions. One solution is finite when its argument is large and the other solution diverges. Thus, if this equation was valid at large R we would have chosen the solution

$$\hat{K}^{(II)} = B R^2 K_\nu(RY \sqrt{h_Y}) \quad (3.20)$$

with some undetermined constant B . Of course, at large R we do not really trust this equation since we are no longer in the region (3.18). However, as discussed above, we expect that at large momentum the dominant contributions to the correlation functions will come from regions with RY which is not very large, and it should not matter what the solution (or the background) looks like at large R . So, we will choose the specific

solution (3.20) in region II, assuming that even if we also include the other solution with some coefficient (which will be present for generic IR resolutions of the background) it will not change the leading large momentum behavior.

Notice that for $Y \gg 1$ there is an overlap between region I (3.15) and region II (3.18). Indeed, if for large Y we look at values of R scaling as

$$R \sim \frac{1}{Y \ln^\gamma(Y)}, \quad \gamma > \frac{1}{2}, \quad (3.21)$$

we are simultaneously in both regions. We would like to exploit this overlap between regions I and II to determine the coefficients $\{p_{0,0}, p_{2\nu,0}, B\}$ by matching $K^{(I)}$ and $K^{(II)}$ in the overlap region. We will treat the cases of non-integer and integer ν separately.

3.2.1 Non-integer ν

From the UV boundary condition (3.14) we have (defining a normalized integration constant C_ν)

$$p_{0,0} \equiv \rho_s^{2-\nu} C_\nu = \rho_s^{2-\nu} (1 + \mathcal{O}(R_{UV}^2 \ln(R_{UV}))). \quad (3.22)$$

Next, comparing the coefficients of terms going as $R^{2n-\nu}$ (for integer $n \geq 0$) in the expansion (3.12) in region I with the expansion of the Bessel function in region II, in the overlap region (3.21), we find

$$\begin{aligned} R^{2n-\nu} B \kappa_{2n,0} Y^{2n-\nu} h_Y^{n-\frac{\nu}{2}} &\simeq R^{2n-\nu} \sum_{m=0}^n p_{2n,m} \ln^m(R) \\ &= R^{2n-\nu} \sum_{m=0}^n p_{2n,m} (\ln(RY) - \ln(Y))^m \\ &= R^{2n-\nu} \left(p_{2n,n} (-1)^n \ln^n(Y) + \mathcal{O}(\ln(RY) \ln^{n-1}(Y)) \right), \end{aligned} \quad (3.23)$$

where in the bottom line we used (3.18). Since we can use the scaling (3.21), we find

$$\begin{aligned} B &= \frac{p_{2n,n}}{\kappa_{2n,0}} (-1)^n Y^{\nu-2n} h_Y^{\frac{\nu}{2}-n} \ln^n(Y) \times \left(1 + \mathcal{O}\left(\frac{\ln(\ln(Y))}{\ln(Y)}\right) \right) \\ &=_{p_{0,0}} \left(Y \sqrt{h_Y} \right)^\nu \times \left(1 + \mathcal{O}\left(\frac{\ln(\ln(Y))}{\ln(Y)}\right) \right). \end{aligned} \quad (3.24)$$

Note that this result is independent of n (which is a consistency check for the validity of both expansions). Similarly, comparing the coefficients of terms going as $R^{2n+\nu}$ gives

$$\begin{aligned}
R^{2n+\nu} B \kappa_{2\nu+2n,0} Y^{2n+\nu} h_Y^{n+\frac{\nu}{2}} &\simeq R^{2n+\nu} \sum_{m=0}^n p_{2n+2\nu,m} \ln^m(R) \\
&= R^{2n+\nu} \sum_{m=0}^n p_{2n+2\nu,m} (\ln(RY) - \ln(Y))^m \\
&= R^{2n+\nu} \left(p_{2n+2\nu,n} (-1)^n \ln^n(Y) + \mathcal{O}(\ln(RY) \ln^{n-1}(Y)) \right),
\end{aligned} \tag{3.25}$$

leading to

$$\begin{aligned}
B &= \frac{p_{2n+2\nu,n}}{\kappa_{2\nu+2n,0}} (-1)^n Y^{-\nu-2n} h_Y^{-\frac{\nu}{2}-n} \ln^n(Y) \times \left(1 + \mathcal{O}\left(\frac{\ln(\ln(Y))}{\ln(Y)}\right) \right) \\
&= -p_{2\nu,0} \frac{2^{2\nu} \Gamma(1+\nu)}{\Gamma(1-\nu)} \left(Y \sqrt{h_Y} \right)^{-\nu} \times \left(1 + \mathcal{O}\left(\frac{\ln(\ln(Y))}{\ln(Y)}\right) \right)
\end{aligned} \tag{3.26}$$

(which, again, is independent of n). Given (3.24) this determines

$$p_{2\nu,0} = -p_{0,0} \frac{\Gamma(1-\nu)}{2^{2\nu} \Gamma(1+\nu)} \left(Y \sqrt{h_Y} \right)^{2\nu} \times \left(1 + \mathcal{O}\left(\frac{\ln(\ln(Y))}{\ln(Y)}\right) \right). \tag{3.27}$$

To summarize, for non-integer ν , matching $K^{(I)}$ and $K^{(II)}$ in the overlap region determines (to leading order at large Y , with corrections of order $\ln(\ln(Y))/\ln(Y)$)

$$\begin{aligned}
p_{0,0} &= \rho_s^{2-\nu} C_\nu, \quad C_\nu = 1 + \mathcal{O}(R_{UV}^2 \ln(R_{UV})), \\
p_{2\nu,0} &= -\rho_s^{2-\nu} \frac{\Gamma(1-\nu)}{2^{2\nu} \Gamma(1+\nu)} \left(Y \sqrt{h_Y} \right)^{2\nu} C_\nu, \\
B &= \rho_s^{2-\nu} \left(Y \sqrt{h_Y} \right)^\nu C_\nu.
\end{aligned} \tag{3.28}$$

3.2.2 Integer ν

Now we tackle the slightly more difficult case of integer ν . Again, from (3.14) we have

$$p_{0,0} = \rho_s^{2-\nu} C_\nu, \quad C_\nu = 1 + \mathcal{O}(R_{UV}^2 \ln(R_{UV})). \tag{3.29}$$

The comparison in the overlap region of terms of order $R^{2n-\nu}$ for $\nu > n \geq 0$ is exactly the same as before. This leads to

$$B = p_{0,0} \left(Y \sqrt{h_Y} \right)^\nu \times \left(1 + \mathcal{O}\left(\frac{\ln(\ln(Y))}{\ln(Y)}\right) \right). \tag{3.30}$$

For $n = \nu$ we find

$$R^\nu B Y^\nu h_Y^{\frac{\nu}{2}} \sum_{m=0}^1 \kappa_{2\nu,m} \ln^m(RY\sqrt{h_Y}) \simeq R^\nu \sum_{m=0}^{\nu+1} p_{2\nu,m} \ln^m(R) \quad (3.31)$$

or

$$R^\nu B Y^\nu h_Y^{\frac{\nu}{2}} \left(\kappa_{2\nu,0} + \frac{1}{2} \kappa_{2\nu,1} \ln(h_Y) + \kappa_{2\nu,1} \ln(RY) \right) \simeq R^\nu \sum_{m=0}^{\nu+1} p_{2\nu,m} (\ln(RY) - \ln(Y))^m. \quad (3.32)$$

The coefficients $p_{2\nu,m}$ with $m > 0$ scale as $Y^{2\nu} p_{0,0}$. Therefore, in the large Y limit, contributions with $0 < m < \nu + 1$ are all smaller than the contribution coming from $p_{2\nu,\nu+1}$. On the other hand, $p_{2\nu,0}$ is an independent coefficient so that apriori we do not know if it is smaller. Thus, to leading order in $\ln(RY)/\ln(Y)$, the right-hand side of (3.32) takes the form

$$R^\nu \left(p_{2\nu,0} + (-1)^{\nu+1} p_{2\nu,\nu+1} \ln^{\nu+1}(Y) \right) \times \left(1 + \mathcal{O}\left(\frac{\ln(RY)}{\ln(Y)}\right) \right). \quad (3.33)$$

Comparing the leading order expansion of the left-hand side of (3.32) with (3.33), we find

$$B Y^\nu h_Y^{\frac{\nu}{2}} \kappa_{2\nu,0} = p_{2\nu,0} + (-1)^{\nu+1} p_{2\nu,\nu+1} \ln^{\nu+1}(Y), \quad (3.34)$$

where we have neglected terms of order $\ln(\ln(Y))/\ln(Y)$. This implies (using our result (3.30) for B) that to leading order in $1/\ln(Y)$

$$p_{2\nu,0} = (-1)^\nu p_{2\nu,\nu+1} \ln^{\nu+1}(Y), \quad (3.35)$$

with $p_{2\nu,\nu+1}$ given in terms of $p_{0,0}$ in table 2.

At this stage all the free parameters $p_{0,0}$, $p_{2\nu,0}$ and B are determined. Thus, for each value of $n > \nu$, matching the leading coefficients at order $R^{2n-\nu}$ in the overlap region must be automatic. We explicitly verified that this is indeed the case.

To summarize, for integer ν a matching of $K^{(I)}$ and $K^{(II)}$ in the overlap region determines (to leading order in $\ln(\ln(Y))/\ln(Y)$)

$$\begin{aligned} p_{0,0} &= \rho_s^{2-\nu} C_\nu, & C_\nu &= 1 + \mathcal{O}(R_{UV}^2 \ln(R_{UV})), \\ p_{2\nu,0} &= (-1)^\nu p_{2\nu,\nu+1} \ln^{\nu+1}(Y) C_\nu, \\ B &= \rho_s^{2-\nu} \left(Y \sqrt{h_Y} \right)^\nu C_\nu. \end{aligned} \quad (3.36)$$

3.2.3 Explicit examples

We give here some explicit examples of KT BtB propagators (with $a = 0$).

For the massless $\nu = 2$ case our result is identical to the result of Krasnitz [21, 22] :

$$\hat{K}(Y, R) = C_2 \left[1 + \frac{1}{4} b Y^2 R^2 \ln(R) + b^2 Y^4 R^4 \left(-\frac{1}{128} \ln(R) + \frac{1}{64} \ln^2(R) - \frac{1}{48} \ln^3(R) - \frac{1}{48} \ln^3 Y + \mathcal{O}(\ln^2(Y) \ln(\ln(Y))) \right) + \mathcal{O}(R^6) \right]. \quad (3.37)$$

For $\nu = 5/2$ we find

$$\hat{K}(Y, R) = C_{5/2} \rho^{-1/2} \left[1 + b Y^2 R^2 \left(-\frac{1}{36} + \frac{1}{6} \ln(R) \right) + b^2 Y^4 R^4 \left(\frac{1}{16} + \frac{1}{18} \ln(R) + \frac{1}{24} \ln^2(R) \right) - \frac{1}{45} Y^5 R^5 \left(h_Y^{5/2} + \mathcal{O}(\ln^2(Y) \ln(\ln(Y))) \right) + \mathcal{O}(R^6) \right]. \quad (3.38)$$

For the tachyonic $\nu = 1$ case we find

$$\hat{K}(Y, R) = C_1 \rho \left[1 + b Y^2 R^2 \left(\frac{1}{4} \ln(R) - \frac{1}{4} \ln^2(R) + \frac{1}{4} \ln^2(Y) + \mathcal{O}(\ln(Y) \ln(\ln(Y))) \right) + \mathcal{O}(R^4) \right]. \quad (3.39)$$

3.3 Two-point functions

In any holographic background the two-point function may be extracted from the UV behavior of the BtB propagator. Two-point functions in AdS were studied in [18, 19], and two-point functions in KT were studied (for $m^2 = 0$) in [21, 22]. One subtlety that was emphasized in [32] is that in order to get correct Ward identities one should use a prescription for evaluating the correlator in which the UV cutoff ρ_{UV} is taken to zero only at the very end of the calculation. This is the prescription we will follow. Alternatively, one may use holographic renormalization [23, 33–42] to calculate the two-point functions. Adding local counterterms does not change the result.

For completeness, we will first describe how to obtain the two-point functions in AdS and then move on to the KT case. The reader may refer to [18, 32] for details. Consider a scalar field in Euclidean AdS space (with a cutoff at $\rho = \rho_{UV}$) with the action

$$S = \frac{1}{2} \int d^{d+1} x \sqrt{g} (g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + m^2 \phi^2). \quad (3.40)$$

Evaluating the action (in momentum space) on a solution $\hat{\phi}$ to the equations of motion gives

$$S = \lim_{\rho \rightarrow \rho_{UV}} \frac{1}{2} \int \frac{1}{\rho^3} \delta(\vec{k} + \vec{q}) \hat{\phi}(\vec{k}) \partial_\rho \hat{\phi}(\vec{q}) d^4 k d^4 q. \quad (3.41)$$

The two-point function of the operator dual to ϕ is given by the second derivative of the action with respect to sources

$$\langle \hat{\mathcal{O}}_\nu(\vec{k}) \hat{\mathcal{O}}_\nu(\vec{q}) \rangle = \delta(\vec{k} + \vec{q}) \frac{1}{\rho_{UV}^3} \rho_{UV}^{2-\nu} \lim_{\rho \rightarrow \rho_{UV}} \partial_\rho \hat{K}_\nu(\vec{q}, \rho). \quad (3.42)$$

This expression may be readily evaluated. We start with integer ν . Here, we have (see (3.9))

$$\begin{aligned} \partial_\rho \hat{K}_\nu(\vec{k}) &= \rho_s^{-1} \partial_R \hat{K}_\nu = \rho_s^{1-\nu} C_\nu \left(\sum_{n=0}^{\nu-1} \kappa_{2n,0} Y^{2n} (2n - \nu + 2) R^{2n-\nu+1} \right. \\ &\quad \left. + \sum_{n=\nu}^{\infty} \sum_{m=0}^1 \kappa_{2n+2\nu,m} Y^{2n+2\nu} ((\nu + 2n + 2) R^{2n+\nu+1} (\ln(RY))^m + m R^{2n+\nu+1}) \right). \end{aligned} \quad (3.43)$$

So, the correlation function is given by the $R_{UV} \rightarrow 0$ limit of

$$\begin{aligned} &\delta(\vec{k} + \vec{q}) \rho_s^{-2\nu} R_{UV}^{-(\nu+1)} \times \\ &\lim_{R \rightarrow R_{UV}} \left\{ \left(\sum_{n=0}^{\nu-1} \kappa_{2n,0} (R_{UV} Y)^{2n} + \sum_{n=0}^{\infty} \sum_{m=0}^1 \kappa_{2n+2\nu,m} (R_{UV} Y)^{2n+2\nu} (\ln(R_{UV} Y))^m \right)^{-1} \right. \\ &\quad \times \left(\sum_{n=0}^{\nu-1} \kappa_{2n,0} Y^{2n} (2n - \nu + 2) R^{2n-\nu+1} \right. \\ &\quad \left. \left. + \sum_{n=0}^{\infty} \sum_{m=0}^1 \kappa_{2n+2\nu,m} Y^{2n+2\nu} ((\nu + 2n + 2) R^{2n+\nu+1} (\ln(RY))^m + m R^{2n+\nu+1}) \right) \right\} \end{aligned} \quad (3.44)$$

where the denominator comes from the normalization C_ν . This expression diverges as $\rho_{UV} \rightarrow 0$, but it is easy to verify that all divergent terms are analytic in k . Analyticity of these divergences implies that they are unphysical contact terms in position space which may be subtracted (by adding appropriate counter-terms). The finite non-analytic terms are given by

$$\langle \hat{\mathcal{O}}_\nu(\vec{k}) \hat{\mathcal{O}}_\nu(\vec{q}) \rangle = \delta(\vec{k} + \vec{q}) \frac{(-1)^{\nu+1}}{2^{2\nu-2} \Gamma(\nu)^2} k^{2\nu} \ln(k \rho_s). \quad (3.45)$$

Despite appearances, this is independent of ρ_s , since the ρ_s -dependent term is analytic. Note that, as stated earlier, if we had first taken the $\rho_{UV} \rightarrow 0$ limit, the numerical coefficient we would have obtained would have been different (and wrong).

The analysis for non-integer ν is very similar, the only difference is that instead of (3.9) we have the expansion (3.10). Since non-analyticity only comes from the second sum, the results are similar,

$$\langle \hat{\mathcal{O}}_\nu(\vec{k}) \hat{\mathcal{O}}_\nu(\vec{q}) \rangle = \delta(\vec{k} + \vec{q}) k^{2\nu} (2\nu) \kappa_{2\nu,0} = \delta(\vec{k} + \vec{q}) \frac{-\Gamma(1-\nu)}{2^{2\nu-1} \Gamma(\nu)} k^{2\nu}. \quad (3.46)$$

To obtain the position space correlation function, we need to Fourier transform the above expressions. We find that for both integer and non-integer ν we have [18, 32]

$$\langle \mathcal{O}_\nu(\vec{x}_1) \mathcal{O}_\nu(\vec{x}_2) \rangle = \frac{2\nu^2(1+\nu)}{\pi^2} \frac{1}{|\vec{x}_1 - \vec{x}_2|^{4+2\nu}}. \quad (3.47)$$

The analysis of the two-point functions in the cascading (KT) background closely follows the AdS case and generalizes the results of [21] to massive fields. We still have

$$\langle \hat{\mathcal{O}}_\nu(\vec{k}) \hat{\mathcal{O}}_\nu(\vec{q}) \rangle = \delta(\vec{k} + \vec{q}) \lim_{\rho \rightarrow \rho_{UV}} \frac{1}{\rho^3} \hat{K}_\nu(\vec{k}) \partial_\rho \hat{K}_\nu(\vec{q}). \quad (3.48)$$

The leading non-analytic contribution to this expression is similar to that in the AdS case because of the form of the power law expansion of the propagator (see (3.12) and (3.13)). We find

$$\langle \hat{\mathcal{O}}_\nu(\vec{k}) \hat{\mathcal{O}}_\nu(\vec{q}) \rangle = \delta(\vec{k} + \vec{q}) \rho_s^{-2-\nu} (2\nu) p_{2\nu,0}. \quad (3.49)$$

For integer ν this gives us

$$\langle \hat{\mathcal{O}}_\nu(\vec{k}) \hat{\mathcal{O}}_\nu(\vec{q}) \rangle = \delta(\vec{k} + \vec{q}) \frac{(-1)^{\nu+1} b^\nu}{2^{2\nu-2} (\nu+1) \Gamma(\nu)^2} k^{2\nu} (\ln(Y))^{\nu+1}, \quad (3.50)$$

while for non-integer ν we find

$$\langle \hat{\mathcal{O}}_\nu(\vec{k}) \hat{\mathcal{O}}_\nu(\vec{q}) \rangle = -\delta(\vec{k} + \vec{q}) \frac{\Gamma(1-\nu)}{2^{2\nu-1} \Gamma(\nu)} k^{2\nu} (b \ln(Y))^\nu. \quad (3.51)$$

By Fourier transforming we may obtain the leading behavior of the short distance correlation function. We find

$$\langle \mathcal{O}_\nu(\vec{x}_1) \mathcal{O}_\nu(\vec{x}_2) \rangle = \frac{2\nu^2(1+\nu)}{\pi^2} \frac{(b \ln(\rho_s/|\vec{x}_1 - \vec{x}_2|))^\nu}{|\vec{x}_1 - \vec{x}_2|^{4+2\nu}} = \frac{2(-b)^\nu \nu^2(1+\nu)}{\pi^2} \frac{(\ln(\Lambda|\vec{x}_1 - \vec{x}_2|))^\nu}{|\vec{x}_1 - \vec{x}_2|^{4+2\nu}} \quad (3.52)$$

for both integer and non-integer ν . Curiously, in the KT case we find that the momentum space two-point functions are not smooth in ν , while in AdS the $\nu \rightarrow n$ limit, with $n \in \mathbb{Z}$, commutes with the Fourier transform. In both cases the position space answers are smooth in ν . Similarly, the full momentum space KT BtB propagator does not seem to have a smooth limit as ν approaches an integer, while its Fourier transform does (at least for the leading terms which we computed).

On general grounds, we expect the correlation functions of the cascading gauge theories to reflect the variation in the rank of the gauge group with the momentum. Since in the large K limit of an $SU(K)$ gauge theory, there is a standard normalization of the operators (which is the one coming from the dual string theory) in which all correlation functions scale as K^2 , we expect to have a normalization in the KT case for which all correlation functions scale as $N_{eff}^2 \sim b^2 \ln^2(k/\Lambda)$ (more precisely, since some factors of $\ln(k)$ disappear when we go to position space for integer ν , we expect the position space answers to scale as $b^2 \ln^2(\Lambda|\vec{x}_i - \vec{x}_j|)$). The two-point functions we found above have this scaling for the massless $\nu = 2$ case, but not for other cases. However, we can always rescale our operators (by a momentum-dependent factor) so that the 2-point functions will all scale as $b^2 \ln^2(\Lambda|\vec{x}_i - \vec{x}_j|)$ as expected. Therefore, we will define normalized operators $\tilde{\mathcal{O}}_\nu(\vec{k}) \equiv \hat{\mathcal{O}}_\nu(\vec{k})/(b \ln(k/\Lambda))^{(\nu-2)/2}$ which obey the expected scaling for their 2-point functions. In the following section we will compute general correlation functions of these normalized operators and see that they scale as N_{eff}^2 . Another natural scaling which is often used is to divide the previous operators by N_{eff} , so that the two-point functions do not scale while higher N -point functions scale as N_{eff}^{2-N} . The operators obeying this scaling are $\tilde{\mathcal{O}}'_\nu(\vec{k}) \equiv \hat{\mathcal{O}}_\nu(\vec{k})/(b \ln(k/\Lambda))^{\nu/2}$.

4 Higher N -point functions

In this section we compute the large-momentum behavior of N -point functions in asymptotically cascading backgrounds. We will focus on the specific contribution to N -point functions coming from a single non-derivative vertex which couples the N fields. We expect that the qualitative features that we find will be present also in other contributions to the N -point functions. We begin by performing our analysis in asymptotically AdS backgrounds, both because as far as we know this computation has not been performed before in momentum space, and because many features of the AdS computation carry over in a straightforward manner to the cascading case.

In general, the expression for the correlators (both in the AdS case and in the cascading case) is quite complicated, involving an integral of Bessel functions which we do not know how to compute exactly. However, we will be able to prove that the results for the non-analytic terms in the correlators are always finite (independent of the UV cutoff), so that the theory is well-defined. In some special cases we will be able to write down a closed-form expression for the leading large momentum behavior of the correlators. Separating our computation into regions I and II as we did in the discussion of the KT BtB propagator, we will show that in some cases (both in AdS and in KT), the region I contribution is dominant at large momentum, and can be explicitly computed.

4.1 Tree level N -point functions in AdS

4.1.1 General expression for the N -point functions

We are interested in computing the contribution to an N -point function of operators dual to scalar fields, coming from a tree-level diagram involving a single interaction vertex (with coefficient λ_N) coupling these scalar fields together. The general rules of computation in AdS space [18, 19] imply that the result in momentum space is given by

$$\mathcal{A}_N = \langle \hat{\mathcal{O}}_1(\vec{k}_1) \dots \hat{\mathcal{O}}_N(\vec{k}_N) \rangle = \delta\left(\sum \vec{k}_i\right) \lambda_N \int \prod_i \hat{K}_i(k_i; \rho) \rho^{-5} d\rho. \quad (4.1)$$

As discussed above, the same expression should be true at high momentum even in spaces which are only asymptotically AdS.

In AdS space we have an explicit result, described above, for the BtB propagator \hat{K} at any value of ρ . However, since we are planning to generalize our results to the KT background, it is natural to separate the contributions to (4.1) as coming from region I and region II, where in region I we use the perturbative expansion of the BtB propagator (3.3)-(3.5), which is useful for $RY \ll 1$, and in region II we use the precise expression involving the Bessel function. These regions are analogous to the two regions we used in our KT computation. In principle, in AdS we could extend region II all the way down to $R = R_{UV}$. However, it will be instructive to choose a separation point R_t obeying $R_t Y \ll 1$ such that the power series expression can be used at $R < R_t$ and the Bessel function expression can be used at $R > R_t$. In the KT case we will be forced to use such a procedure.

In region I of AdS we have seen that the BtB propagator is given by

$$\hat{K}_\nu^{(I)} = \rho_s^{2-\nu} C_\nu R^{-\nu+2} \left(\sum_{n=0}^{\nu-1} \kappa_{2n,0} (RY)^{2n} + \sum_{n=0}^{\infty} \sum_{m=0}^1 \kappa_{2n+2\nu,m} (RY)^{2n+2\nu} \ln^m(RY) \right) \quad (4.2)$$

for integer ν , and by

$$\hat{K}_\nu^{(I)} = \rho_s^{2-\nu} C_\nu Y^\nu R^2 \left((RY)^{-\nu} \sum_{n=0}^{\infty} \kappa_{2n,0} (RY)^{2n} + (RY)^\nu \sum_{n=0}^{\infty} \kappa_{2\nu+2n,0} (RY)^{2n} \right) \quad (4.3)$$

for non-integer ν . The coefficients κ are given in table 1. In order to unify our expressions for integer and non-integer values of ν , we will write both cases as

$$\hat{K}_\nu^{(I)} = \rho_s^{2-\nu} C_\nu R^{-\nu+2} \sum_{n,m,s} \kappa_{2n+2\nu s,m} (RY)^{2n+2\nu s} \ln^m(RY). \quad (4.4)$$

In the sum, n , m and s take the following values. $s \in \{0,1\}$ distinguishes the first and second terms in (4.2) and (4.3). If ν is an integer and $s = 1$ then $m \in \{0,1\}$. Otherwise, $m = 0$. Finally, n goes from zero to infinity except when ν is an integer and $s = 0$, in which case it goes from zero to $\nu - 1$.

Ignoring for now the momentum conserving δ -function, the tree level N -point function may be written as

$$\begin{aligned} \mathcal{A}_N &= \lambda_N \rho_s^{-4} \int_{R_{UV}}^{\infty} dR R^{-5} \prod_{i=1}^N \hat{K}_i(Y_i, R) = \mathcal{A}_N^{(I)} + \mathcal{A}_N^{(II)} \\ &= \lambda_N \rho_s^{-4} \int_{R_{UV}}^{R_t} dR R^{-5} \prod_{i=1}^N \hat{K}_i^{(I)}(Y_i, R) + \lambda_N \rho_s^{-4} \int_{R_t}^{\infty} dR R^{-5} \prod_{i=1}^N \hat{K}_i^{(II)}(Y_i, R). \end{aligned} \quad (4.5)$$

We do not know how to perform the integral over the Bessel functions in region II. However, in region I we can perform the integral explicitly:

$$\begin{aligned} \mathcal{A}_N^{(I)} &= \rho_s^{-4} \lambda_N \prod_{i=1}^N C_{\nu_i} \sum_{\{n_i, m_i, s_i\}} \left(\prod_{i=1}^N \rho_s^{2-\nu_i} Y_i^{2n_i+2s_i\nu_i} \kappa_{2n_i+2s_i\nu_i, m_i} \right) \\ &\quad \times \int_{R_{UV}}^{R_t} \frac{dR}{R} R^{\bar{n}} \prod_{j=1}^N \ln^{m_j}(RY_j) \end{aligned} \quad (4.6)$$

where the summation is over $\{n_i, m_i, s_i\}$ in the range described above, and we define

$$\bar{n} \equiv -4 + 2N - \sum_{i=1}^N \nu_i + 2 \sum_{i=1}^N \nu_i s_i + 2 \sum_{i=1}^N n_i. \quad (4.7)$$

In order to evaluate the integral in (4.6), we arrange the indices such that all the m_i 's for which $m_i \neq 0$ appear first, at $i = 1, \dots, \bar{m}$ where $\bar{m} \equiv \sum_{i=1}^N m_i$. We note that \bar{m} counts the number of $\ln(Y_i)$ contributions from integer ν terms. For $\bar{n} \neq 0$ we find that the integral (4.6) is given by

$$\int \prod_{i=1}^{\bar{m}} \ln(Y_i R) R^{\bar{n}} \frac{dR}{R} = R^{\bar{n}} \sum_{k=0}^{\bar{m}} \frac{(-1)^k}{\bar{n}^{k+1}} \sum_{\substack{j_1, \dots, j_k \\ j_1 \neq \dots \neq j_k}}^{\bar{m}} \left(\prod_{i \neq j_1, \dots, j_k} \ln(Y_i R) \right), \quad (4.8)$$

where the second sum on the right is given by one when $k = 0$. Thus, the contribution of these terms is given by:

$$\begin{aligned} \mathcal{A}_N^{(I)} = \rho_s^{-4} \lambda_N \prod_{i=1}^N C_{\nu_i} \sum_{\{n_i, m_i, s_i\}, \bar{n} \neq 0} \left(\prod_{i=1}^N \rho_s^{2-\nu_i} Y_i^{2n_i+2s_i\nu_i} \kappa_{2n_i+2s_i\nu_i, m_i} \right) \\ \times R^{\bar{n}} \sum_{k=0}^{\bar{m}} \frac{(-1)^k}{\bar{n}^{k+1}} \sum_{\substack{j_1, \dots, j_k \\ j_1 \neq \dots \neq j_k}}^{\bar{m}} \left(\prod_{i \neq j_1, \dots, j_k} \ln(Y_i R) \right) \Bigg|_{R_{UV}}^{R_t}. \end{aligned} \quad (4.9)$$

For the special $\bar{n} = 0$ case, we find

$$\int \prod_{i=1}^{\bar{m}} \ln(Y_i R) \frac{dR}{R} = \sum_{t=0}^{\bar{m}} \frac{(\ln(\mu R))^{t+1}}{(t+1)!} \partial^{(t)} P(-\ln(\mu)), \quad (4.10)$$

where we have defined $P(x) \equiv \prod_{i=1}^{\bar{m}} (\ln(Y_i) + x)$, $\partial^{(t)}$ is the t 'th derivative, and μ is an arbitrary integration constant which should be independent of the momenta Y_i . Therefore, the $\bar{n} = 0$ terms contribute

$$\begin{aligned} \mathcal{A}_N^{(I)} = \rho_s^{-4} \lambda_N \prod_{i=1}^N C_{\nu_i} \sum_{\{n_i, m_i, s_i\}, \bar{n}=0} \left(\prod_{i=1}^N \rho_s^{2-\nu_i} Y_i^{2n_i+2s_i\nu_i} \kappa_{2n_i+2s_i\nu_i, m_i} \right) \\ \times \sum_{t=0}^{\bar{m}} \frac{(\ln(\mu R))^{t+1}}{(t+1)!} \partial^{(t)} P(-\ln(\mu)) \Bigg|_{R_{UV}}^{R_t}. \end{aligned} \quad (4.11)$$

4.1.2 Locality of UV divergences and the $R_{UV} \rightarrow 0$ limit

Some of the terms in the integrals we wrote over region I (the ones with $\bar{n} \leq 0$) are divergent as $R_{UV} \rightarrow 0$. The correlation functions on AdS that we have been computing (which can be, for example, those of the Klebanov-Witten supersymmetric gauge theory [26]) should be renormalizable. Therefore, the divergences in (4.6) arising from the

$R_{UV} \rightarrow 0$ limit must be non-analytic in at most $(N - 2)$ different momenta⁵. This means that the divergences do not contribute to the correlation functions in position space at generic separated points and that they can be canceled by local counter-terms. In our expressions for the correlation function in region I, non-analytic contributions in k_i (or in Y_i) appear only when the corresponding $s_i = 1$ (such contributions are non-analytic due to the non-integer powers of k_i^2 in the non-integer ν case, and due to the $\ln(Y_i)$ in the integer ν case). Thus, we should require that (4.6) converges at the lower limit of integration whenever at least $(N - 1)$ of the s_i are equal to one. The most stringent condition comes from the case when a single $s_r = 0$; we require that the corresponding value of \bar{n} must be positive (for any choice of n_i)

$$-4 + 2N + \sum_{i=1}^N \nu_i - 2\nu_r > 0, \quad \nu_r \in \{\nu_1, \dots, \nu_N\}. \quad (4.12)$$

Introducing

$$\nu_{max} \equiv \max\{\nu_1, \dots, \nu_N\}, \quad \nu_{tot} \equiv \sum_{i=1}^N \nu_i, \quad (4.13)$$

we conclude that for the theory to be renormalizable, λ_N must vanish whenever

$$-4 + 2N + \nu_{tot} - 2\nu_{max} \leq 0. \quad (4.14)$$

We can rewrite this condition in terms of the dimensions $\Delta_i = \nu_i + 2$ of the dual gauge theory operators. In this language we find that λ_N must vanish if

$$\frac{1}{2} \sum_{i=1}^N \Delta_i \leq \max\{\Delta_1, \dots, \Delta_N\}, \quad \text{or if for some } j \quad \Delta_j \geq \sum_{i=1, i \neq j}^N \Delta_i. \quad (4.15)$$

This is, indeed, a well-known condition for renormalizability also from the position-space analysis of AdS correlators [43]. The case with an equality in (4.15) is called the extremal correlator case, and the bulk couplings λ_N must vanish in this case as well.

The condition described above holds in all known AdS backgrounds. In particular, it holds for the KK modes in the Klebanov-Witten background. We will see in the next subsection that precisely when this condition holds, the correlators of the same operators in the cascading KT gauge theory are also finite. In this sense the renormalizability of the KT N -point correlation functions is linked to the renormalizability of the corresponding correlators in the conformal Klebanov-Witten gauge theory.

⁵All such divergences can then be subtracted by introducing local counter-terms of the type needed to renormalize all $\{2, \dots, N - 1\}$ -point correlation functions.

We also note that the C_{ν_i} 's will not contribute to N -point functions with $N \geq 3$, which implies that one may take the $\rho_{UV} \rightarrow 0$ limit before evaluating the correlator (as shown in [32]). This follows from the fact that $\left(\prod_{i=1}^N C_{\nu_i}\right) = 1 + \mathcal{O}(R_{UV}^2)$ so that a non-trivial contribution from the C_{ν_i} may survive in the $R_{UV} \rightarrow 0$ limit only if there is a non-analytic term going as a negative power of R_{UV} . As we have just shown, such divergent terms do not exist. Therefore, when extracting the non-analytic contributions to the N -point functions with $N \geq 3$ we may set $\prod_i C_{\nu_i} = 1$.

4.1.3 Analysis of leading terms

Evaluating the contribution to the correlation function (4.5) from region II is technically difficult. However, we will show that there are some N -point correlators which are dominated by the $\bar{n} = 0$ terms in the series expansion in region I. We will evaluate these terms explicitly below. For more general correlators the best expression we have is (4.5).

We are interested in computing the correlation functions in the large momentum limit. For simplicity, we assume that all momenta k_i are of the same order. This allows us to introduce a typical momentum scale $k_* = Y_*/\rho_s$ with

$$Y_* = \frac{1}{N} \sum_{i=1}^N Y_i. \quad (4.16)$$

When we perform an expansion of our expressions at large Y_* , we can ignore terms of order $\ln(Y_i/Y_j)$ or $\ln(Y_i/Y_*)$ compared to terms of order $\ln(Y_*)$.

In the large momentum limit, with $Y_i \sim Y_j$, we choose the separation, R_t , between regions I and II to be

$$R_t = \frac{1}{Y_* \ln^\gamma(Y_*)}, \quad \gamma > 0, \quad (4.17)$$

such that $R_t Y_i \ll 1$ (namely, R_t is in both regions I and II). We note that (4.16) and (4.17) imply that $\ln(Y_*)$ and $\ln(R_t)$ are non-analytic in all momenta. There is some freedom in choosing R_t and Y_* . However, in the large momentum expansion, the final expressions we find will depend on the choice of R_t only through subleading terms. The specific choice above for R_t is motivated by the fact that in some exact computations (such as a four-point correlation function which we will present below) it correctly gives some of the subleading terms as well.

We would like to find correlation functions whose major contribution to non-analytic terms at high momenta is from region I, where we can evaluate the integrals explicitly.

We note that at leading order in Y_* , the region II contribution is equal to

$$\mathcal{A}^{II} = \rho_s^{2N-4-\nu_{tot}} \lambda_N \int_{R_t}^{\infty} R^{2N-4} (Y_*)^{\nu_{tot}} \prod_{i=1}^N K_{\nu_i}(RY_*) \frac{dR}{R} \quad (4.18)$$

(up to a constant depending on the ratios of the momenta, which we assume to be finite in the large momentum limit). Consider the case $\nu_{tot} > 2(N-2)$ ⁶. Since $R_t Y_*$ is small in the limit we are interested in, and the small R behavior of the integrand goes as $R^{2N-4-\nu_{tot}-1}$, the above integral is divergent as the lower bound goes to zero. Hence, we conclude that it is dominated by the contribution from the lower bound, which is of order

$$\mathcal{A}^{II} = \rho_s^{2N-4-\nu_{tot}} \lambda_N \int_{R_t} \frac{dR}{R} R^{2N-4-\nu_{tot}} \sim \rho_s^{2N-4-\nu_{tot}} \lambda_N R_t^{2N-4-\nu_{tot}}. \quad (4.19)$$

We would like to compare this expression with the non-analytic contributions from region I, which we computed in §4.1.1. We start with the $\bar{n} > 0$ contributions. Approximating $Y_i \sim Y_*$ in (4.9) and requiring that it dominate over the region II contribution gives us the condition

$$\frac{(\gamma \ln(\ln(Y_*)))^{\bar{m}}}{(\ln(Y_*))^{\gamma(2\sum \nu_i s_i + 2\sum n_i)}} = \frac{(\gamma \ln(\ln(Y_*)))^{\bar{m}}}{(\ln(Y_*))^{\gamma(\bar{n} + \nu_{tot} - 2(N-2))}} \gg 1. \quad (4.20)$$

Obviously this does not hold in the large momentum limit for any $\gamma > 0$.

Therefore, only terms with $\bar{n} = 0$ in region I (4.11) may dominate. We saw that the contributions from the lower bound of the integral are always analytic and thus, uninteresting. However, in the particular case of $\bar{n} = 0$, the contribution to (4.11) from the upper region of integration is non-analytic for any values of s_i , since it always contains a $\ln(R_t) \sim -\ln(Y_*)$ term. Recall that we are interested in the leading non-analytic contribution at large momentum. For integer ν_i , every non-vanishing value of s_i produces a power of $\ln(Y_i)$ (from the term with $m_i = 1$), so we would like to have as many non-zero values of the s_i corresponding to integer ν_i 's as possible. From the analysis of the previous subsection, we know that there can be at most $(N-2)$ s_i 's which do not vanish (for $\bar{n} = 0$).

We find that the condition for the contribution from the upper bound of integration

⁶If $\nu_{tot} \leq 2(N-2)$ then it can be shown that region II will always dominate over region I, and so, we are not interested in this case.

of region I to dominate over the region II result is

$$Y_\star^{4-2N+\nu_{tot}} \sum_{t=0}^{\bar{m}} \left(\ln \left(\frac{\mu}{Y_\star \ln^\gamma(Y_\star)} \right) \right)^{t+1} \partial^{(t)} \prod_{i=1}^{\bar{m}} (\ln(Y_i) + x) \Big|_{x=-\ln(\mu)} \gg \frac{1}{(Y_\star \ln^\gamma(Y_\star))^{2N-4-\nu_{tot}}}, \quad (4.21)$$

or

$$\sum_{t=0}^{\bar{m}} \left(\ln \left(\frac{\mu}{Y_\star \ln^\gamma(Y_\star)} \right) \right)^{t+1} \partial^{(t)} \prod_{i=1}^{\bar{m}} (\ln(Y_i) + x) \Big|_{x=-\ln(\mu)} \gg (\ln(Y_\star))^{\gamma(4-2N+\nu_{tot})}. \quad (4.22)$$

This is satisfied at large momenta provided that

$$\bar{m} + 1 > \gamma(4 - 2N + \nu_{tot}). \quad (4.23)$$

We can always choose a small enough γ (which must also satisfy (4.17), $\gamma > 0$) so that this inequality is satisfied.

To summarize, in order for an N -point function to be dominated for large momentum by the region I integral, we need that two constraints be satisfied. One is a constraint on γ which will make the contribution of the $\bar{n} = 0$ term in region I dominate over region II (4.23). It may always be satisfied. The other constraint is that a $\bar{n} = 0$ term should exist; there should exist a choice of s_i and n_i such that

$$-4 + 2N + \nu_{tot} + 2\left(\sum s_i \nu_i - \nu_{tot}\right) + 2n_{tot} = 0, \quad (4.24)$$

with $n_{tot} \equiv \sum_i n_i$, recalling that we must also have

$$-4 + 2N + \nu_{tot} - 2\nu_{max} > 0. \quad (4.25)$$

Such a choice does not exist for all correlation functions that we want to compute (for example for generic non-integer values of ν_i). However, in many cases such a choice does exist, and for any N one can find some large enough ν_i such that this constraint is satisfied.

4.1.4 Examples of correlation functions

We may now evaluate explicitly the correlation functions which are dominated by the $\bar{n} = 0$ term in region I. These are correlation functions which satisfy (4.25), and which have contributions which satisfy (4.24). In this case we find that, to leading order in

the momenta,

$$\begin{aligned} \langle \hat{\mathcal{O}}_1(\vec{k}_1) \cdots \hat{\mathcal{O}}_N(\vec{k}_N) \rangle &= \delta(\sum \vec{k}_i) \lambda_N \sum_{\{n_i, s_i \in S\}} \left(\prod_{i=1}^N \kappa_{2n_i+2s_i\nu_i, m_i} \right) \left(\prod_{i=1}^N k_i^{2n_i+2s_i\nu_i} \right) \\ &\times \sum_{t=0}^{\bar{m}} \frac{(-\ln(k_*/\Lambda))^{t+1}}{(t+1)!} \partial^{(t)} \prod_{\substack{i=1 \\ s_i=1}}^{\bar{m}} (\ln(k_i/\Lambda) + x) \Big|_{x=0}, \end{aligned} \quad (4.26)$$

where S is the set of all s_i 's and n_i 's which satisfy (4.24), and we take $m_i = 1$ whenever ν_i is integer and $s_i = 1$. The constants κ appear in table 1. We have used our freedom of choosing $\mu = 1/(\Lambda\rho_s)$ to rewrite the logarithms using an arbitrary mass scale Λ . The choice of μ does not affect any non-analytic terms in the results since there are no μ -dependent terms which are non-analytic. When all the ν_i are non-integer⁷ (and also in other cases with all $m_i = 0$), the second line is simply given by $-\ln(k_*/\Lambda)$.

As an example consider 3-point correlation functions. Here we can have at most a single $s_i \neq 0$. These correlators will have a leading term which we can compute if there exist integer n_i 's such that

$$2 \sum_{i=1}^3 n_i = \sum_{i=1}^3 \nu_i - 2\nu_j - 2 \quad (4.27)$$

when $s_j = 1$ for some j , or

$$2 \sum_{i=1}^3 n_i = \sum_{i=1}^3 \nu_i - 2 \quad (4.28)$$

when all $s_i = 0$. In both cases we must also have

$$0 < 2 + \sum_{i=1}^3 \nu_i - 2\nu_{max}. \quad (4.29)$$

Defining $m_j = 1$ if ν_j is an integer, and zero otherwise, we find from the (4.27) terms

$$\begin{aligned} \langle \hat{\mathcal{O}}_1(\vec{k}_1) \cdots \hat{\mathcal{O}}_3(\vec{k}_3) \rangle_a &= \delta(\vec{k}_1 + \vec{k}_2 + \vec{k}_3) \lambda_N \sum_{j=1}^3 \sum_{\{n_i\} \in S_j} \\ &\times \left(\prod_{i=1}^3 k_i^{2n_i} \right) k_j^{2\nu_j} (-\ln(k_*/\Lambda)) \left(\ln(k_j/\Lambda) - \frac{1}{2}(\ln(k_*/\Lambda)) \right)^{m_j} \\ &\left(\prod_{i=1}^3 \frac{(-1)^{n_i} \Gamma(\nu_i - n_i)}{2^{2n_i} \Gamma(n_i + 1) \Gamma(\nu_i)} \right) \left(\frac{(-1)^{\nu_j} 2}{\Gamma(1 - \nu_j) \Gamma(\nu_j)} \right)^{m_j} \left(\frac{-\Gamma(n_j - \nu_j + 1)}{2^{2\nu_j} \Gamma(n_j + \nu_j + 1)} \right), \end{aligned} \quad (4.30)$$

⁷Note that the Klebanov-Witten background, unlike the $AdS_5 \times S^5$ background, has KK modes with non-integer values of the ν_i .

where S_j are all the n_i 's which satisfy (4.27). Note that for integer ν_j the expression (4.30) contains a ratio of diverging Gamma functions. This should be understood as the finite limit of the ratio when ν_j approaches the corresponding integer. From the (4.28) terms we find a contribution of

$$\langle \hat{\mathcal{O}}_1(\vec{k}_1) \cdots \hat{\mathcal{O}}_3(\vec{k}_3) \rangle_b = \delta(\vec{k}_1 + \vec{k}_2 + \vec{k}_3) \lambda_N \sum_{\{n_i\} \in S} \left(\prod_{i=1}^3 \frac{(-1)^{n_i} \Gamma(\nu_i - n_i)}{2^{2n_i} \Gamma(n_i + 1) \Gamma(\nu_i)} k_i^{2n_i} \right) \left(-\ln\left(\frac{k_\star}{\Lambda}\right) \right), \quad (4.31)$$

where here S are all the combinations of $\{n_i\}$ which satisfy (4.28). The correlation function is generally given by

$$\langle \hat{\mathcal{O}}_1(\vec{k}_1) \cdots \hat{\mathcal{O}}_3(\vec{k}_3) \rangle = \langle \hat{\mathcal{O}}_1(\vec{k}_1) \cdots \hat{\mathcal{O}}_3(\vec{k}_3) \rangle_a + \langle \hat{\mathcal{O}}_1(\vec{k}_1) \cdots \hat{\mathcal{O}}_3(\vec{k}_3) \rangle_b. \quad (4.32)$$

The expression (4.32) for the 3-point correlation function should be understood as the leading non-analytic contribution for fixed values of ν_i , in the limit $\frac{k_i}{\Lambda} \rightarrow \infty$, $\frac{k_\star}{\Lambda} \rightarrow \infty$. For specific choices⁸ of ν_i 's, the first contribution in (4.32), *i.e.* (4.30), dominates. In this case the contribution (4.31) is subdominant, and it is inconsistent to keep it along with (4.30). It is only when (4.30) and (4.31) are of the same order (in the large momentum limit) that the 3-point correlation function is given by a sum (4.32).

As a specific example, the three point massless ($\nu_i = 2$) correlator in momentum space is dominated at large momentum by the terms with $s_j = 1$ ($j = 1, 2, 3$),

$$\langle \hat{\mathcal{O}}_2(\vec{k}_1) \cdots \hat{\mathcal{O}}_2(\vec{k}_3) \rangle = \delta(\vec{k}_1 + \vec{k}_2 + \vec{k}_3) \frac{\lambda_N}{16} \sum_{j=1}^3 k_j^4 \left(\ln(k_\star/\Lambda) \ln(k_j/\Lambda) - \frac{1}{2} (\ln(k_\star/\Lambda))^2 \right). \quad (4.33)$$

As a test of our methods we can look at the four-point function of operators with indices $\nu_1 = \nu_2 = 5/2$ and $\nu_3 = \nu_4 = 1/2$. In the half-integer ν case, modified Bessel functions of the second kind are exponents multiplied by polynomials, and so the exact momentum-space correlation function (4.5) can be evaluated explicitly. From this explicit computation one finds that the leading non-contact terms are given (up to the overall delta function) by

$$\frac{1}{6} (k_1^2 + k_2^2 - 3(k_3 + k_4)^2) \ln((k_1 + k_2 + k_3 + k_4)/\Lambda). \quad (4.34)$$

Our equation (4.26) gives

$$\frac{1}{6} (k_1^2 + k_2^2 - 3(k_3 + k_4)^2) \ln(k_\star/\Lambda), \quad (4.35)$$

⁸This occurs in particular when (4.27) can be solved for integer values of ν_i 's.

so that the leading large momentum non-analytic terms are indeed identical.

Finally, we would like to emphasize that the general result (4.32) is obtained in the $\frac{k_i}{\Lambda} \rightarrow \infty, \frac{k_*}{\Lambda} \rightarrow \infty$ limit with ν_j kept fixed. However, one can not use (4.32) to compute the 3-point correlation functions arising in the limit as some ν_i approach integer values. In fact, in the limit $\nu_i \rightarrow \hat{\nu}_i$ for integer values of $\hat{\nu}_i$ satisfying (4.27), one has

$$\lim_{\nu_i \rightarrow \hat{\nu}_i} \langle \hat{\mathcal{O}}_{\nu_1}(\vec{k}_1) \cdots \hat{\mathcal{O}}_{\nu_3}(\vec{k}_3) \rangle \neq \langle \hat{\mathcal{O}}_{\hat{\nu}_1}(\vec{k}_1) \cdots \hat{\mathcal{O}}_{\hat{\nu}_3}(\vec{k}_3) \rangle. \quad (4.36)$$

The reason for the apparent discrepancy is that the limit of the short distance behavior of the correlation functions does not commute with the limit in which dimensions of some operators approach integer values. At the technical level, if the limit $\nu_i \rightarrow \hat{\nu}_i$ is taken before the $\frac{k_i}{\Lambda} \rightarrow \infty$ limit, certain $\bar{n} \neq 0$ terms in (4.6) can dominate and produce a leading non-analytic behavior. As we explicitly demonstrate in appendix B, the contribution of these $\bar{n} \neq 0$ terms in the limit $\nu_i \rightarrow \hat{\nu}_i$ precisely reproduces the dominant $\bar{n} = 0$ contribution of the $\hat{\mathcal{O}}_{\hat{\nu}_i}(\vec{k}_i)$ correlators in the large k_i limit. Thus, the correlation functions do have a smooth $\nu_i \rightarrow \text{integer}$ limit, as expected from the fact that the position space propagators in AdS have a smooth limit. However, this smooth limit is not evident in our expressions above.

4.2 Tree level N -point functions in asymptotically cascading geometries

4.2.1 General expression for the N -point functions

We would like to repeat the same analysis in asymptotically cascading geometries. The bulk-to-boundary propagator in region I is given by

$$\hat{K}_\nu^{(I)} = R^{-\nu+2} \left(\sum_{n=0}^{\nu-1} \sum_{m=0}^n p_{2n,m} R^{2n} \ln^m(R) + \sum_{n=0}^{\infty} \sum_{m=0}^{n+\nu+1} p_{2n+2\nu,m} R^{2n+2\nu} \ln^m(R) \right) \quad (4.37)$$

for integer ν , and by

$$\hat{K}_\nu^{(I)} = R^{-\nu+2} \left(\sum_{n=0}^{\infty} \sum_{m=0}^n p_{2n,m} R^{2n} \ln^m(R) + \sum_{n=0}^{\infty} \sum_{m=0}^n p_{2\nu+2n,m} R^{2n+2\nu} \ln^m(R) \right) \quad (4.38)$$

for non-integer ν . The relevant coefficients, p , are given in table 2. Again, we can write both cases as

$$\hat{K}_\nu^{(I)} = R^{-\nu+2} \sum_{n,m,s} p_{2n+2\nu s,m} R^{2n+2\nu s} \ln^m(R), \quad (4.39)$$

where the only difference from the AdS case is in the range of m (and in the precise coefficients).

Up to the momentum conservation δ -function, the tree level N -point function arising from an N -point vertex in the bulk is given by

$$\begin{aligned}\mathcal{A}_N &= \lambda_N \rho_s^{-4} \int_{R_{UV}}^{\infty} dR R^{-5} \prod_{i=1}^N \hat{K}_i(Y_i, R) = \mathcal{A}_N^{(I)} + \mathcal{A}_N^{(II)} \\ &= \lambda_N \rho_s^{-4} \int_{R_{UV}}^{R_t} dR R^{-5} \prod_{i=1}^N \hat{K}_i^{(I)}(Y_i, R) + \lambda_N \rho_s^{-4} \int_{R_t}^{\infty} dR R^{-5} \prod_{i=1}^N \hat{K}_i^{(II)}(Y_i, R),\end{aligned}\tag{4.40}$$

where R_t is chosen to be in the overlap between regions I and II, described in (3.21). We are taking the background to be the KT background all the way to $R \rightarrow \infty$ even though this is singular, since we expect (and will verify) that the leading contributions at large momentum come from small values of R , for which the large R behavior of the background is irrelevant. In region I we have

$$\mathcal{A}_N^{(I)} = \rho_s^{-4} \lambda_N \sum_{\{n_i, m_i, s_i\}} \left(\prod_{i=1}^N p_{2n_i+2s_i\nu_i, m_i} \right) \int_{R_{UV}}^{R_t} \frac{dR}{R} R^{\bar{n}} \prod_{j=1}^N \ln^{m_j}(R),\tag{4.41}$$

where we again define

$$\bar{n} \equiv -4 + 2N - \sum_{i=1}^N \nu_i + 2 \sum_{i=1}^N \nu_i s_i + 2 \sum_{i=1}^N n_i.\tag{4.42}$$

Using (3.36) and (3.28), we note the following properties of the coefficients $p_{a,b}$ which give the leading large Y contributions. For non-integer ν

$$p_{2n,n} = C_\nu \rho_s^{2-\nu} (-b)^n Y^{2n} \kappa_{2n,0},\tag{4.43}$$

$$p_{2\nu+2n,n} = C_\nu \rho_s^{2-\nu} Y^{2n+2\nu} (-b)^n (h_Y)^\nu \kappa_{2\nu+2n,0},\tag{4.44}$$

while for integer ν

$$p_{2n,n} = C_\nu \rho_s^{2-\nu} (-b)^n Y^{2n} \kappa_{2n,0}, \quad n < \nu\tag{4.45}$$

$$p_{2n,n+1} = C_\nu \rho_s^{2-\nu} (-b)^n Y^{2n} (\nu+1)^{-1} \kappa_{2n,1}, \quad n > \nu\tag{4.46}$$

$$p_{2\nu+2s,s} = C_\nu \rho_s^{2-\nu} (-1)^s (bY^2)^{\nu+s} (\ln(Y))^{\nu+1} (\nu+1)^{-1} \kappa_{2\nu+2s,1}\tag{4.47}$$

$$= (-1)^\nu (\ln(Y))^{\nu+1} p_{2\nu+2s,\nu+s+1}.\tag{4.48}$$

Now, we wish to evaluate the leading contributions to (4.41) at large momentum, which come from the terms with the most logarithmic contributions. Recalling that $0 \leq m_i \leq n_i + \sigma_i(\nu_i + 1)$ (where $\sigma_i \equiv s_i$ for ν_i integer and zero otherwise), leading logarithms in (4.41) always come from factors related to the coefficients (4.43)-(4.48). Thus we can rewrite the leading contributions to (4.41) as

$$\begin{aligned} \mathcal{A}_N^{(I)} &\sim \rho_s^{-4} \lambda_N \sum_{\{n_i, s_i\}} \\ &\times \left(\prod_{i=1}^N C_{\nu_i} \rho_s^{2-\nu_i} (-b)^{n_i+\nu_i\sigma_i} Y_i^{2n_i+2s_i\nu_i} (b \ln(Y_i))^{(s_i-\sigma_i)\nu_i} (\nu_i+1)^{-\sigma_i} \kappa_{2n_i+2s_i\nu_i, \sigma_i} \right) \\ &\times \int_{R_{UV}}^{R_t} \frac{dR}{R} R^{\bar{n}} (\ln(R))^{n_{tot}} \prod_{i=1}^N ((-1)^{\nu_i} (\ln(Y_i))^{\nu_i+1} + (\ln(R))^{\nu_i+1})^{\sigma_i}. \end{aligned} \quad (4.49)$$

To match with the earlier notation, we define $\bar{m} \equiv \sum \sigma_i$, and as in the AdS case we rearrange the indices so that the first \bar{m} indices specify the $\sigma_i \neq 0$ contributions. We need to evaluate the integral

$$\int R^{\bar{n}} (\ln(R))^{n_{tot}} \prod_{i=1}^{\bar{m}} ((-1)^{\nu_i} (\ln(Y_i))^{\nu_i+1} + (\ln(R))^{\nu_i+1}) \frac{dR}{R}. \quad (4.50)$$

We start with the $\bar{n} \neq 0$ case. It will be convenient to use the following identity :

$$((-1)^\nu (\ln(Y))^{\nu+1} + (\ln(R))^{\nu+1}) = \sum_{t=0}^{\nu} (-1)^t \binom{\nu+1}{t} (\ln(YR))^{\nu+1-t} (\ln(Y))^t. \quad (4.51)$$

The integral we wish to evaluate can be written as

$$\int R^{\bar{n}} (\ln(R))^{n_{tot}} \prod_{i=1}^{\bar{m}} \left(\sum_{t=0}^{\nu_i} (-1)^t \binom{\nu_i+1}{t} (\ln(Y_i R))^{\nu_i+1-t} (\ln(Y_i))^t \right) \frac{dR}{R}. \quad (4.52)$$

In order to evaluate the leading term in (4.52), we only need to keep track of the terms with the largest power of $(\ln(Y_i))$ (the $t = \nu_i$ term). To see this, note that from (4.8) we find

$$\begin{aligned} &\int R^{\bar{n}} (\ln(R))^{n_{tot}} (\ln(Y_1 R)) \dots (\ln(Y_n R)) \frac{dR}{R} \\ &= R^{\bar{n}} (\ln(R))^{n_{tot}} \sum_{k=0}^n (-1)^k \frac{1}{\bar{n}^{k+1}} \sum_{\substack{j_1 \dots j_k \\ j_1 \neq \dots \neq j_k}}^n \left(\prod_{i \neq j_1, \dots, j_k} \ln(Y_i R) \right) \\ &\quad + \mathcal{O} \left(R^{\bar{n}} (\ln(R))^{n_{tot}-1} \prod \ln(Y_i R) \right). \end{aligned} \quad (4.53)$$

Thus,

$$\begin{aligned}
& \int R^{\bar{n}} (\ln(R))^{n_{tot}} \prod_{i=1}^{\bar{m}} ((-1)^{\nu_i} (\ln(Y_i))^{\nu_i+1} + (\ln(R))^{\nu_i+1}) \frac{dR}{R} \\
&= R^{\bar{n}} (\ln(R))^{n_{tot}} \left(\prod_{i=1}^{\bar{m}} (-1)^{\nu_i} (\nu_i + 1) (\ln(Y_i))^{\nu_i} \right) \sum_{k=0}^{\bar{m}} \frac{(-1)^k}{\bar{n}^{k+1}} \sum_{\substack{j_1 \dots j_k \\ j_1 \neq \dots \neq j_k}}^{\bar{m}} \left(\prod_{i \neq j_1, \dots, j_k} \ln(Y_i R) \right) \\
&\quad + \mathcal{O} \left(R^{\bar{n}} (\ln(R))^{n_{tot}-1} \prod_{i=1}^{\bar{m}} (\ln(Y_i))^{\nu_i} \prod \ln(Y_i R) \right). \quad (4.54)
\end{aligned}$$

This gives us a leading contribution of the form

$$\begin{aligned}
\mathcal{A}_N^{(I)} &\sim \rho_s^{-4} \lambda_N \prod_{i=1}^N C_{\nu_i} \sum_{\{n_i, s_i\}} \left(\prod_{i=1}^N \rho_s^{2-\nu_i} Y_i^{2n_i+2s_i\nu_i} (-b \ln(R))^{n_i} (b \ln(Y_i))^{s_i\nu_i} \kappa_{2n_i+2s_i\nu_i, \sigma_i} \right) \\
&\quad \times R^{\bar{n}} \sum_{k=0}^{\bar{m}} \frac{(-1)^k}{\bar{n}^{k+1}} \sum_{\substack{j_1 \dots j_k \\ j_1 \neq \dots \neq j_k}}^{\bar{m}} \left(\prod_{i \neq j_1, \dots, j_k} \ln(Y_i R) \right) \Bigg|_{R_{UV}}^{R_t} \quad (4.55)
\end{aligned}$$

which differs from its AdS counterpart (4.9)

$$\begin{aligned}
\mathcal{A}_N^{(I)} &= \rho_s^{-4} \lambda_N \prod_{i=1}^N C_{\nu_i} \sum_{\{n_i, m_i, s_i\}} \left(\prod_{i=1}^N \rho_s^{2-\nu_i} Y_i^{2n_i+2s_i\nu_i} \kappa_{2n_i+2s_i\nu_i, m_i} \right) \\
&\quad \times R^{\bar{n}} \sum_{k=0}^{\bar{m}} \frac{(-1)^k}{\bar{n}^{k+1}} \sum_{\substack{j_1, \dots, j_k \\ j_1 \neq \dots \neq j_k}}^{\bar{m}} \left(\prod_{i \neq j_1, \dots, j_k} \ln(Y_i R) \right) \Bigg|_{R_{UV}}^{R_t} \quad (4.56)
\end{aligned}$$

only by some powers of logs (as we found in the two-point functions).

The analysis of the $\bar{n} = 0$ contributions in KT is again very similar to that of AdS. Using (4.10) we find that the $\bar{n} = 0$ term contributes

$$\begin{aligned}
\mathcal{A}_N^{(I)} &\sim \rho_s^{-4} \lambda_N \sum_{\{n_i, s_i\}, \bar{n}=0} \sum_{t=0}^{n_{tot} + \sum_{i=1}^{\bar{m}} (\nu_i+1)} \frac{(\ln(\mu R))^{t+1}}{(t+1)!} \partial^{(t)} P(-\ln(\mu)) \Bigg|_{R_{UV}}^{R_t} \\
&\quad \times \left(\prod_{i=1}^N C_{\nu_i} \rho_s^{2-\nu_i} (-b)^{n_i+\nu_i\sigma_i} Y_i^{2n_i+2s_i\nu_i} (b \ln(Y_i))^{(s_i-\sigma_i)\nu_i} (\nu_i+1)^{-\sigma_i} \kappa_{2n_i+2s_i\nu_i, \sigma_i} \right), \quad (4.57)
\end{aligned}$$

where here

$$P(x) = x^{n_{tot}} \prod_{i=1}^{\bar{m}} ((-1)^{\nu_i} (\ln(Y_i))^{\nu_i+1} + x^{\nu_i+1}). \quad (4.58)$$

The AdS counterpart of this expression is given in (4.11).

4.2.2 The $R_{UV} \rightarrow 0$ limit

We will now show that if the ν_i satisfy (4.25), which is the case whenever the λ_N do not vanish (since we are using the same couplings as we had in the AdS case and we assume that the AdS case is renormalizable), then the divergences in (4.41) coming from the $R_{UV} \rightarrow 0$ limit are non-analytic in at most $(N - 2)$ momenta, so that they correspond to contact terms. Indeed, the first time a term non-analytic in $(N - 1)$ momenta appears as $R_{UV} \rightarrow 0$ is when $n_i = 0$ for all i , and a single $s_r = 0$. In this case

$$\bar{n} = -4 + 2N + \sum_{i=1}^N \nu_i - 2\nu_r \geq -4 + 2N + \nu_{tot} - 2\nu_{max}. \quad (4.59)$$

The right-hand side of (4.59) must be strictly positive for $\lambda_N \neq 0$, so such a term is independent of R_{UV} in the $R_{UV} \rightarrow 0$ limit. Thus, whenever λ_N is non-vanishing, divergent terms in (4.41) as $R_{UV} \rightarrow 0$ are non-analytic in at most $(N - 2)$ momenta. This implies that a cascading version of a conformal field theory is holographically renormalizable whenever the original conformal field theory is. By the same arguments as for the AdS case, we also find that the $\prod C_{\nu_i}$ do not contribute to non-analytic terms, and we will ignore them from here on.

4.2.3 Analysis of leading terms

As in the AdS case, we wish to consider N -point correlators which are dominated by the region I contribution $\mathcal{A}_N^{(I)}$.

Again, we introduce a typical momentum $k_* = Y_*/\rho_s$ with

$$Y_* \equiv \frac{1}{N} \sum_{i=1}^N Y_i, \quad (4.60)$$

and choose the separation between regions I and II to be at

$$R_t = \frac{1}{Y_* \ln^\gamma(Y_*)}, \quad \gamma > \frac{1}{2}, \quad (4.61)$$

so that R_t is in the overlap of regions I and II for all momenta, provided they are not vastly different, $Y_i \sim Y_j$. By a computation similar to the AdS case, we find that the region II contribution is dominated by⁹

$$\mathcal{A}^{II} \sim \lambda_N \rho_s^{2N-4-\nu_{tot}} R_t^{2N-4-\nu_{tot}}. \quad (4.62)$$

⁹As in AdS, we consider $\nu_{tot} > 2(N - 2)$, as this is the only case where region I will turn out to dominate over region II.

Comparing this to the region I contribution with $\bar{n} > 0$, we find that region I dominates whenever

$$\frac{(\gamma \ln(\ln(Y_*)))^{\bar{m}}}{\ln(Y_*)^{(2\gamma-1)(\sum s_i \nu_i + \sum n_i)}} \gg 1 \quad (4.63)$$

which is always false, meaning that only terms with $\bar{n} = 0$ may contribute if region I is to dominate over region II. Since μ is independent of the momenta, we find that the condition for the $\bar{n} = 0$ term in region I to dominate over region II is

$$n_{tot} + \sum_i s_i \nu_i + \bar{m} + 1 > \gamma(4 + \nu_{tot} - 2N), \quad (4.64)$$

implying (using $\bar{n} = 2N - 4 - \nu_{tot} + 2\sum \nu_i s_i + 2n_{tot} = 0$)

$$\bar{m} + 1 > \left(\gamma - \frac{1}{2}\right)(4 + \nu_{tot} - 2N). \quad (4.65)$$

Again, this may always be satisfied for an appropriate choice of $\gamma > \frac{1}{2}$. The constraints we find for the existence of the $\bar{n} = 0$ term are thus the same as those for the AdS case, (4.24) and (4.25).

For the special case of the three-point function with equal integer ν , we find that the leading term has $\bar{m} = 1$, so that (4.65) reduces to

$$\frac{1}{2} \frac{3\nu + 2}{3\nu - 2} > \gamma, \quad (4.66)$$

which is consistent with $\gamma > \frac{1}{2}$.

4.2.4 Leading expressions for correlation functions

As in AdS, we may evaluate explicitly the correlation functions which are dominated by the $\bar{n} = 0$ term in region I. These are correlation functions for which there exists a choice of n_i and s_i such that

$$-4 + 2N + \nu_{tot} + 2\left(\sum_i s_i \nu_i - \nu_{tot}\right) + 2n_{tot} = 0. \quad (4.67)$$

In this case, we find that to leading order in the large momentum limit

$$\begin{aligned}
\langle \hat{\mathcal{O}}_1(\vec{k}_1) \cdots \hat{\mathcal{O}}_N(\vec{k}_N) \rangle &\sim \delta(\sum \vec{k}_i) \lambda_N \sum_{\{n_i, s_i \in S\}} \left(\prod_{i=1}^N \kappa_{2n_i+2s_i\nu_i, \sigma_i} \right) \\
&\times \left(\prod_{i=1}^N k_i^{2n_i+2s_i\nu_i} (b \ln(k_i/\Lambda))^{(s_i-\sigma_i)\nu_i} (-b)^{n_i+\nu_i\sigma_i} (\nu_i+1)^{-\sigma_i} \right) \\
&\times \sum_{t=0}^{n_{tot}+\sum_{i=1}^{\bar{m}}(\nu_i+1)} \frac{(-\ln(k_*/\Lambda))^{t+1}}{(t+1)!} \partial^{(t)} \left\{ x^{n_{tot}} \prod_{i=1}^{\bar{m}} ((-1)^{\nu_i} (\ln(k_i/\Lambda))^{\nu_i+1} + x^{\nu_i+1}) \right\} \Big|_{x=0},
\end{aligned} \tag{4.68}$$

where we have set $\mu = 1$ in order to write the solutions using the natural scale $\Lambda = 1/\rho_s$. Notice that any apparent μ -dependence in expressions of the form (4.68) is only through analytic terms which disappear in position space.

To simplify this expression, we observe that if $t < n_{tot}$ the second sum will vanish. Thus, (4.68) can be rewritten as

$$\begin{aligned}
\langle \hat{\mathcal{O}}_1(\vec{k}_1) \cdots \hat{\mathcal{O}}_N(\vec{k}_N) \rangle &\sim \delta(\sum \vec{k}_i) \lambda_N \sum_{\{n_i, s_i \in S\}} \left(\prod_{i=1}^N \kappa_{2n_i+2s_i\nu_i, \sigma_i} \right) \\
&\times \left(\prod_{i=1}^N k_i^{2n_i+2s_i\nu_i} b^{s_i\nu_i} (\ln(k_i/\Lambda))^{(s_i-\sigma_i)\nu_i} \right) (b \ln(k_*/\Lambda))^{n_{tot}} n_{tot}! \\
&\times \sum_{t=0}^{\sum_{i=1}^{\bar{m}}(\nu_i+1)} \frac{(-\ln(k_*/\Lambda))^{t+1}}{(n_{tot}+t+1)!} \partial^{(t)} \left\{ \prod_{i=1}^{\bar{m}} \left(\frac{(\ln(k_i/\Lambda))^{\nu_i+1} + (-1)^{\nu_i} x^{\nu_i+1}}{\nu_i+1} \right) \right\} \Big|_{x=0}.
\end{aligned} \tag{4.69}$$

For three-point functions we again find that there are two types of contributions. Those from the (4.27) term are given by

$$\begin{aligned}
\langle \hat{\mathcal{O}}_1(\vec{k}_1) \cdots \hat{\mathcal{O}}_3(\vec{k}_3) \rangle_a &= \delta(\vec{k}_1 + \vec{k}_2 + \vec{k}_3) \lambda_N \sum_{j=1}^3 \sum_{\{n_i\} \in S_j} \\
&\times \left(\prod_{i=1}^3 (b k_i^2 \ln(k_*/\Lambda))^{n_i} \right) k_j^{2\nu_j} b^{\nu_j} n_{tot}! \left(\frac{-\ln(k_*/\Lambda) (\ln(k_j/\Lambda))^{\nu_j+1}}{(n_{tot}+1)! (\nu_j+1)} + \frac{(\ln(k_*/\Lambda))^{\nu_j+2} \nu_j!}{(n_{tot}+\nu_j+2)!} \right) \\
&\times \left(\prod_{i=1}^3 \frac{(-1)^{n_i} \Gamma(\nu_i - n_i)}{2^{2n_i} \Gamma(n_i+1) \Gamma(\nu_i)} \right) \left(\frac{(-1)^{\nu_j} 2}{\Gamma(1-\nu_j) \Gamma(\nu_j)} \right)^{\sigma_j} \left(\frac{-\Gamma(n_j - \nu_j + 1)}{2^{2\nu_j} \Gamma(n_j + \nu_j + 1)} \right),
\end{aligned} \tag{4.70}$$

where we use the same notation as in (4.30). The other contributions come from the

(4.28) term

$$\begin{aligned} \langle \hat{\mathcal{O}}_1(\vec{k}_1) \cdots \hat{\mathcal{O}}_3(\vec{k}_3) \rangle_b &= \delta(\vec{k}_1 + \vec{k}_2 + \vec{k}_3) \lambda_N \\ &\times \sum_{j=1}^3 \sum_{\{n_i\} \in S_j} \left(\prod_{i=1}^3 (b k_i^2 \ln(k_\star/\Lambda))^{n_i} \right) \frac{(-\ln(k_\star/\Lambda))}{n_{tot} + 1} \left(\prod_{i=1}^3 \frac{(-1)^{n_i} \Gamma(\nu_i - n_i)}{2^{2n_i} \Gamma(n_i + 1) \Gamma(\nu_i)} \right). \end{aligned} \quad (4.71)$$

The complete three-point function is given by

$$\langle \hat{\mathcal{O}}_1(\vec{k}_1) \cdots \hat{\mathcal{O}}_3(\vec{k}_3) \rangle = \langle \hat{\mathcal{O}}_1(\vec{k}_1) \cdots \hat{\mathcal{O}}_3(\vec{k}_3) \rangle_a + \langle \hat{\mathcal{O}}_1(\vec{k}_1) \cdots \hat{\mathcal{O}}_3(\vec{k}_3) \rangle_b \quad (4.72)$$

and should be understood in the same sense as the corresponding AdS three-point correlation function (4.32).

As a specific example, the three-point function for the massless $\nu_i = 2$ modes is given by

$$\begin{aligned} \langle \hat{\mathcal{O}}_2(\vec{k}_1) \cdots \hat{\mathcal{O}}_2(\vec{k}_3) \rangle &= \delta(\vec{k}_1 + \vec{k}_2 + \vec{k}_3) \lambda_N \times \\ &\frac{b^2}{48} \sum_{j=1}^3 k_j^4 \left((\ln(k_\star/\Lambda)) (\ln(k_j/\Lambda))^3 - \frac{1}{4} (\ln(k_\star/\Lambda))^4 \right). \end{aligned} \quad (4.73)$$

As we did in the AdS case, we also consider the specific four-point function with $\nu_1 = \nu_2 = 5/2$ and $\nu_3 = \nu_4 = 1/2$. We find that the dominant contribution is of the form

$$\begin{aligned} \frac{b}{12} \left(k_1^2 (\ln(k_1/\Lambda)) + k_2^2 (\ln(k_2/\Lambda)) - 3k_3^2 (\ln(k_3/\Lambda)) - 3k_4^2 (\ln(k_4/\Lambda)) \right. \\ \left. - 12k_3 (\ln(k_3/\Lambda))^{1/2} k_4 (\ln(k_4/\Lambda))^{1/2} \right) \ln(k_\star/\Lambda) \end{aligned} \quad (4.74)$$

which may be compared to the exact AdS result (4.34).

Unlike the AdS correlation functions which have a smooth $\nu_i \rightarrow \text{integer}$ limit, correlation functions in asymptotically cascading geometries do not have such a smooth limit in momentum space. This can be traced to the fact that unlike the AdS case, BtB propagators in asymptotically cascading geometries do not have a smooth $\nu_i \rightarrow \text{integer}$ limit in momentum space. However, since the BtB propagator in position space does have a smooth limit (at least for the leading terms which we computed), we believe that higher order N -point correlation functions in KT are smooth in ν in position space as well.

Finally, note that the overall powers of momentum and of logarithms of momentum that we find in the KT correlators are always given (at leading order) by replacing $k \rightarrow k\sqrt{b\ln(k/\Lambda)}$ in the AdS correlators, although the precise coefficients are different (as are the precise momenta appearing in the logs, but this is something that we are not sensitive to in our leading order computations). This allows us to easily verify that the normalized correlation functions indeed depend on N_{eff} as we expect. In AdS, dimensional arguments imply that a correlator $\langle \hat{\mathcal{O}}_{\nu_1} \cdots \hat{\mathcal{O}}_{\nu_N} \rangle$ scales as $k^{\nu_{tot}-2N+4}$, up to the overall delta function, and sometimes up to logarithmic factors which disappear when we transform to position space. This means that when we normalize the correlation function by dividing by the norms (the square roots of the two-point functions) of the operators, the correlator scales as k^{4-2N} . According to the relation we found above between the KT and AdS results, this implies that the normalized correlator in KT, namely the correlator of the operators which we denoted by $\tilde{\mathcal{O}}'_\nu$ in §3, scales as $k^{4-2N}(b\ln(k/\Lambda))^{2-N}$ (up to the delta function of the momenta, and sometimes up to additional logs which are the same in KT and in AdS and which disappear when we Fourier transform to position space). We expect normalized correlation functions in a large N_{eff} $SU(N_{eff})$ gauge theory to scale as N_{eff}^{2-N} , so the result we find is consistent with the identification $N_{eff}(k) \propto b\ln(k/\Lambda)$. Of course, this is not surprising; in standard gravity computations in AdS, the fact that all tree-level correlators scale as N^2 (in some normalization) comes from the fact that we can normalize the gravity action such that N^2 sits in front of the action. Similarly, in the KT case we can normalize the action so that $N_{eff}^2(\rho)$ sits in front of the gravity action at the scale ρ , leading to the dependence on N_{eff} that we found.

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A Coefficients in the expansion of the KT BtB propagator

We wish to find a perturbative solution for the BtB propagator of KK modes in the asymptotically KT background (2.1). Up to integration constants and an overall power of R^2 , the BtB propagator is given by the solution to (3.11),

$$R^2\psi'' + R\psi' - (\nu^2 + Y^2R^2h(R))\psi = 0. \quad (\text{A.1})$$

We consider the case of $\nu > 0$. The perturbative solution crucially depends on whether ν is an integer or not. We treat these two cases separately.

A.1 Non-integer $\nu > 0$

To find a perturbative solution, we write the series expansion of the field ψ as either

$$\psi(R) = R^{-\nu} \sum_{\substack{n=0 \\ m \leq n}}^{\infty} p_{2n,m} R^{2n} \ln^m(R) \quad (\text{A.2})$$

or

$$\psi(R) = R^{\nu} \sum_{\substack{n=0 \\ m \leq n}}^{\infty} p_{2n+2\nu,m} R^{2n} \ln^m(R). \quad (\text{A.3})$$

Plugging (A.2) into the equation of motion (A.1), we find

$$\sum p_{2n,m} R^{2n-\nu} \left((m-1)m \ln^{m-2}(R) + 2m(2n-\nu) \ln^{m-1}(R) + 4n(n-\nu) \ln^m(R) + \right. \\ \left. R^2 Y^2 (-a \ln^m(R) + b \ln^{m+1}(R)) \right) = 0 \quad (\text{A.4})$$

which may be rewritten as

$$\begin{aligned}
& \sum_{n=0}^{\infty} \sum_{m=0}^n p_{2n,m} 4n(n-\nu) R^{2n-\nu} \ln^m(R) \\
& - \sum_{n=1}^{\infty} \sum_{m=0}^{n-1} p_{2(n-1),m} Y^2 a R^{2n-\nu} \ln^m(R) \\
& + \sum_{n=1}^{\infty} \sum_{m=1}^n p_{2(n-1),m-1} Y^2 b R^{2n-\nu} \ln^m(R) \\
& + \sum_{n=1}^{\infty} \sum_{m=0}^{n-1} p_{2n,m+1} 2(m+1)(2n-\nu) R^{2n-\nu} \ln^m(R) \\
& + \sum_{n=2}^{\infty} \sum_{m=0}^{n-2} p_{2n,m+2} (m+2)(m+1) R^{2n-\nu} \ln^m(R) = 0.
\end{aligned} \tag{A.5}$$

We consider first the leading $\ln^m(R)$ coefficients in the series expansion (A.2). From (A.5) we find that $p_{0,0}$ is arbitrary and

$$p_{2n,n} = \frac{(-bY^2)}{4n(n-\nu)} p_{2(n-1),n-1} = \frac{(-bY^2)^n \Gamma(1-\nu)}{2^{2n} \Gamma(n+1) \Gamma(n+1-\nu)} p_{0,0}, \quad n \geq 1. \tag{A.6}$$

Additionally, for any n and $n > m \geq 0$

$$p_{2n,m} \propto p_{2n,n} \propto (bY^2)^n p_{0,0}. \tag{A.7}$$

A similar analysis for (A.3) leads to arbitrary $p_{2\nu,0}$ and

$$p_{2n+2\nu,n} = \frac{(-bY^2)}{4n(n+\nu)} p_{2(n-1)+2\nu,n-1} = \frac{(-bY^2)^n \Gamma(1+\nu)}{2^{2n} \Gamma(n+1) \Gamma(n+1+\nu)} p_{2\nu,0}, \quad n \geq 1 \tag{A.8}$$

$$p_{2n+2\nu,m} \propto p_{2n+2\nu,n} \propto (bY^2)^n p_{2\nu,0} \tag{A.9}$$

for $n > m \geq 0$.

A.2 Integer $\nu \geq 1$

To find a perturbative solution, we write the series expansion of the field ψ as

$$\psi(R) = R^{-\nu} \left\{ \sum_{n=0}^{\nu-1} \sum_{m=0}^n p_{2n,m} R^{2n} \ln^m(R) + \sum_{n=\nu}^{\infty} \sum_{m=0}^{n+1} p_{2n,m} R^{2n} \ln^m(R) \right\}. \tag{A.10}$$

Plugging (A.10) into the equation of motion (A.1), we find

$$\begin{aligned}
& \sum p_{2n,m} R^{2n-\nu} ((m-1)m \ln^{m-2}(R) + 2m(2n-\nu) \ln^{m-1}(R) + 4n(n-\nu) \ln^m(R) + \\
& R^2 Y^2 (-a \ln^m(R) + b \ln^{m+1}(R))) = 0, \tag{A.11}
\end{aligned}$$

which may be rewritten as

$$\begin{aligned}
& \sum_{n=0}^{\nu-1} \sum_{m=0}^n p_{2n,m} 4n(n-\nu) R^{2n-\nu} \ln^m(R) + \sum_{n=\nu}^{\infty} \sum_{m=0}^{n+1} p_{2n,m} 4n(n-\nu) R^{2n-\nu} \ln^m(R) \\
& + \sum_{n=1}^{\nu} \sum_{m=0}^{n-1} -p_{2(n-1),m} Y^2 a R^{2n-\nu} \ln^m(R) + \sum_{n=\nu+1}^{\infty} \sum_{m=0}^n -p_{2(n-1),m} Y^2 a R^{2n-\nu} \ln^m(R) \\
& + \sum_{n=1}^{\nu} \sum_{m=1}^n p_{2(n-1),m-1} Y^2 b R^{2n-\nu} \ln^m(R) + \sum_{n=\nu+1}^{\infty} \sum_{m=1}^{n+1} p_{2(n-1),m-1} Y^2 b R^{2n-\nu} \ln^m(R) \\
& + \sum_{n=1}^{\nu-1} \sum_{m=0}^{n-1} p_{2n,m+1} 2(m+1)(2n-\nu) R^{2n-\nu} \ln^m(R) \\
& + \sum_{n=\nu}^{\infty} \sum_{m=0}^n p_{2n,m+1} 2(m+1)(2n-\nu) R^{2n-\nu} \ln^m(R) \\
& + \sum_{n=2}^{\nu-1} \sum_{m=0}^{n-2} p_{2n,m+2} (m+2)(m+1) R^{2n-\nu} \ln^m(R) \\
& + \sum_{n=\nu}^{\infty} \sum_{m=0}^{n-1} p_{2n,m+2} (m+2)(m+1) R^{2n-\nu} \ln^m(R) = 0.
\end{aligned} \tag{A.12}$$

From (A.12) we find that $p_{0,0}$ and $p_{2\nu,0}$ are arbitrary and

■ For $1 \leq n < \nu$

$$p_{2n,n} = \frac{(-bY^2)}{4n(n-\nu)} p_{2(n-1),n-1} = \frac{(bY^2)^n \Gamma(\nu-n)}{2^{2n} \Gamma(n+1) \Gamma(\nu)} p_{0,0}, \tag{A.13}$$

with

$$p_{2n,m} \propto p_{2n,n} \propto (bY^2)^n p_{0,0} \tag{A.14}$$

for $n > m \geq 0$.

■ For $n = \nu$

$$p_{2\nu,\nu+1} = \frac{(-bY^2)}{2\nu(\nu+1)} p_{2(\nu-1),\nu-1} = \frac{-(bY^2)^\nu}{2^{2\nu-1} \Gamma(\nu) \Gamma(\nu+2)} p_{0,0}. \tag{A.15}$$

■ For $n > \nu$

$$\begin{aligned}
p_{2n,n+1} &= \frac{(-bY^2)}{4n(n-\nu)} p_{2(n-1),n} = \frac{(-bY^2)^{n-\nu} \Gamma(\nu+1)}{2^{2(n-\nu)} \Gamma(n-\nu+1) \Gamma(n+1)} p_{2\nu,\nu+1} \\
&= \frac{(-1)^{n-\nu+1} \nu (bY^2)^n}{2^{2n-1} \Gamma(n+1) \Gamma(\nu+2) \Gamma(n-\nu+1)} p_{0,0}.
\end{aligned} \tag{A.16}$$

We also note from (A.12), that for a given s , the highest power of m for which an $R^{\nu+2s} \ln^m(R)$ term will have a coefficient depending on $p_{2\nu,0}$ is at $m = s$, in which case we find

$$p_{2\nu+2s,s} = Cp_{0,0} + \frac{(-bY^2)^s \Gamma(\nu+1)}{2^{2s} \Gamma(s+1) \Gamma(s+\nu+1)} p_{2\nu,0}, \quad (\text{A.17})$$

where C is a constant which will not be important in our calculations.

B Correlation functions when some ν_i approach integers

In this section we consider AdS correlation functions with index $\nu_i = \hat{\nu}_i + \delta$ (for fixed integers $\hat{\nu}_i$), in the $\delta \rightarrow 0$ limit. We will show explicitly that this limit does not commute with the large momentum limit. That is, there are terms in the correlation function which are not dominant at large momentum when the ν_i 's are non-integer, but that do become dominant if we first take the $\nu_i \rightarrow \hat{\nu}_i$ limit. Moreover, the leading non-analytic expression we extract for integer-indexed correlation functions by this method agrees with our direct method of calculation (4.26).

We will explicitly discuss a specific limit of three-point functions¹⁰. We consider a set of integer $\hat{\nu}_i$'s and \hat{n}_i 's such that

$$2 \sum_{i=1}^3 \hat{n}_i = \sum_{i=1}^3 \hat{\nu}_i - 2\hat{\nu}_j - 2. \quad (\text{B.1})$$

Following (4.26) and (4.30), the leading non-analytic contribution to the correlation function corresponding to (B.1) takes the form

$$\begin{aligned} \langle \hat{\mathcal{O}}_1(\vec{k}_1) \cdots \hat{\mathcal{O}}_3(\vec{k}_3) \rangle_a &= \lambda_3 \sum_{j=1}^3 \sum_{\{n_i \in S_j\}} \left(\prod_{i=1}^3 (bk_i^2)^{\hat{n}_i} \right) (bk_j^2)^{\hat{\nu}_j} \\ &\times (-\ln(k_*/\Lambda)) \left(\ln(k_j/\Lambda) - \frac{1}{2}(\ln(k_*/\Lambda)) \right) \left(\prod_{i=1}^3 \kappa_{2\hat{n}_i,0} \right) \frac{\kappa_{2\hat{n}_j+2\hat{\nu}_j,1}}{\kappa_{2\hat{n}_j,0}}. \end{aligned} \quad (\text{B.2})$$

We will concentrate only on the contribution of the j 'th element of the above sum.

Next, we consider a small deformation of $\hat{\nu}_j$ of the form

$$\hat{\nu}_j \rightarrow \nu_j = \hat{\nu}_j + \delta_j, \quad \delta_j \ll 1. \quad (\text{B.3})$$

We would like to compare the $\delta_j \rightarrow 0$ limit of such a correlation function to (B.2). Clearly, (B.3) has no $\bar{n} = 0$ terms since it violates the condition (B.1). Therefore, region

¹⁰Other limits can be analyzed similarly and lead to the same conclusion.

I does not dominate, and at finite δ_j one can not evaluate the leading contribution to the correlation functions by our methods. To obtain the leading contribution to the (j 'th component of the) correlation function in the $\delta_j \rightarrow 0$ limit, we need to consider the $\bar{n} \propto \delta_j$ terms. These terms will dominate in the $\delta_j \rightarrow 0$ limit.

We find that there are two important $\bar{n} \propto \delta_j$ terms. The first one is given by

$$\bar{n}_1 = -4 + 6 - \sum_{i=1}^3 \hat{\nu}_i - \delta_j + 2(\hat{\nu}_j + \delta_j) + 2 \sum_{i=1}^3 \hat{n}_i = \delta_j, \quad (\text{B.4})$$

where \bar{n} is the overall power of R , defined in (4.7), and in the last equality we used (B.1). The second term comes from the case where all the s_i 's are set to zero, but \hat{n}_j is replaced by n_j

$$\hat{n}_j \rightarrow n_j = \hat{n}_j + \hat{\nu}_j \quad (\text{B.5})$$

resulting in

$$\bar{n}_2 = -4 + 6 - \sum_{i=1}^3 \hat{\nu}_i - \delta_j + 2 \left(\sum_{i=1}^3 \hat{n}_i + \hat{\nu}_j \right) = -\delta_j. \quad (\text{B.6})$$

We will argue shortly that the other $\bar{n} \propto \delta_j$ terms will be subdominant at large momentum.

The \bar{n}_1 term contributes (see (4.6))

$$\mathcal{A}_{3,1}^{(I)} = \left(\prod_{i=1}^3 (bk_i^2)^{\hat{n}_i} \right) (bk_j^2)^{\hat{\nu}_j} \rho_s^{-\delta_j} Y_j^{2\delta_j} \left(\prod_{i=1}^3 \kappa_{2\hat{n}_i+2\cdot 0\cdot \hat{\nu}_i,0} \right) \frac{\kappa_{2\hat{n}_j+2(\hat{\nu}_j+\delta_j),0}}{\kappa_{2\hat{n}_j+2\cdot 0\cdot \hat{\nu}_j,0}} \frac{(R_t)^{\delta_j}}{\delta_j}, \quad (\text{B.7})$$

while the \bar{n}_2 term contributes

$$\mathcal{A}_{3,2}^{(I)} = \left(\prod_{i=1}^3 (bk_i^2)^{\hat{n}_i} \right) (bk_j^2)^{\hat{\nu}_j} \rho_s^{-\delta_j} \left(\prod_{i=1}^3 \kappa_{2\hat{n}_i+2\cdot 0\cdot \hat{\nu}_i,0} \right) \frac{\kappa_{2(\hat{n}_j+\hat{\nu}_j)+2\cdot 0\cdot (\hat{\nu}_j+\delta_j),0}}{\kappa_{2\hat{n}_j+2\cdot 0\cdot \hat{\nu}_j,0}} \frac{(R_t)^{-\delta_j}}{(-\delta_j)}. \quad (\text{B.8})$$

We first note that taking the $\delta \rightarrow 0$ limit results in

$$\kappa_{2n+2(\hat{\nu}+\delta),0} \rightarrow \frac{1}{2\delta} \kappa_{2n+2\hat{\nu},1} + \mathcal{O}(1), \quad (\text{B.9})$$

$$\kappa_{2n,0} \rightarrow -\frac{1}{2\delta} \kappa_{2n+2\hat{\nu},1} + \mathcal{O}(1), \quad \hat{\nu} + \delta < n. \quad (\text{B.10})$$

Therefore, the non-analytic contributions to $\mathcal{A}_{3,2}^{(I)}$ and $\mathcal{A}_{3,1}^{(I)}$ are of the form

$$\mathcal{A}_{3,1}^{(I)} = \left(\prod_{i=1}^3 (bk_i^2)^{\hat{n}_i} \right) (bk_j^2)^{\hat{\nu}_j} \left(\prod_{i=1}^3 \kappa_{2\hat{n}_i,0} \right) \frac{\kappa_{2\hat{n}_j+2\hat{\nu}_j,1}}{2\delta_j \kappa_{2\hat{n}_j,0}} \quad (\text{B.11})$$

$$\times \frac{1}{\delta_j} \left(-\ln(k_*)\delta_j + \frac{1}{2} (-4\ln(k_j)\ln(k_*) + \ln^2(k_*))\delta_j^2 \right) + \mathcal{O}(\delta^0 \ln(k_*), \delta), \quad (\text{B.12})$$

$$\mathcal{A}_{3,2}^{(I)} = \left(\prod_{i=1}^3 (bk_i^2)^{\hat{n}_i} \right) (bk_j^2)^{\hat{\nu}_j} \left(\prod_{i=1}^3 \kappa_{2\hat{n}_i,0} \right) \frac{-\kappa_{2\hat{n}_j+2\hat{\nu}_j,1}}{2\delta_j \kappa_{2\hat{n}_j,0}} \quad (\text{B.13})$$

$$\times \frac{-1}{\delta_j} \left(\ln(k_*)\delta_j + \frac{1}{2} \ln^2(k_*)\delta_j^2 \right) + \mathcal{O}(\delta^0 \ln(k_*), \delta). \quad (\text{B.14})$$

One can now easily take the $\delta_j \rightarrow 0$ limit to obtain (B.2). From this analysis, it is clear that the other $\bar{n} \propto \delta_j$ terms will be of order δ_j^{-1} , and so will only contribute to order $\ln(k_*)$.

Similarly, it can be verified that the deformation (B.3) for a set of integers satisfying

$$2 \sum_{i=1}^3 \hat{n}_i = \sum_{i=1}^3 \hat{\nu}_i - 2 \quad (\text{B.15})$$

precisely reproduces in the $\delta_j \rightarrow 0$ limit the leading non-analytic contribution (4.31).

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